Polynomial Chaos based design of Robust Input Shapers

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A probabilistic approach which exploits the domain and distribution of the uncertain model parameters has been developed for the design of robust input shapers. A Polynomial Chaos expansion based approach is used to approximate uncertain system states and cost functions in the stochastic space. Residual energy of the system is used as the cost function to design robust input shapers for precise rest-to-rest maneuvers. An optimization problem which minimizes any moment or combination of moments of the distribution function of the residual energy is formulated. Numerical examples are used to illustrate the benefit of using the Polynomial Chaos based probabilistic approach for the determination of robust Input Shapers for uncertain linear systems. The solution of Polynomial Chaos based approach is compared to the minimax optimization based robust input shaper design approach which emulates a Monte Carlo process.

I. Introduction

Precise post-to-point control is of interest in a variety of applications including, scanning probe microscopes, hard disk drives, gantry cranes, flexible arm robots etc. All of these systems are characterized by under-damped modes which are excited by the actuators. Precise position control mandates that the energy in the vibratory modes be dissipated by the end of the maneuver.1–3 These are challenging demands on the controller when the model parameters are known precisely. However, uncertainties in model parameters are ubiquitous and these uncertainties manifest themselves as a deterioration in the performance of the controller. Input Shaping is a technique which shapes the reference command to the system so as to eliminate or minimize the residual energy.4 The first solution to desensitize the Input Shaper to modeling uncertainties was to impose constraints on the sensitivity of the residual energy to model parameter errors to be zero at the end of the maneuver. The resulting solution is referred to as the ZVD Input Shaper.1 The ZVD Input Shaper improves the performance of the pre-filter in the proximity of the nominal model of the system. To exploit knowledge of the domain of uncertainty, the EI Input Shaper5 and the minimax Input Shaper6 were proposed which are worst case design, i.e., they minimize the worst performance of the system over the domain of uncertainty. The minimax Input Shaper requires sampling of the uncertain space which results in a computationally expensive design as the dimension of the uncertain parameter space increases.

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A technique to incorporate the distribution of uncertainty in the Input Shaper design process was proposed by Chang et al.\(^7\) where the expected value of the residual vibration was minimized. The minimax Input Shaper\(^6\) incorporated the probability distribution function as a weighting scheme to differentially weigh the plants that are sampled in the domain of uncertainty. Tenne and Singh\(^8\) proposed the use of unscented transformation to map the Gaussian distributed uncertain parameters into the residual energy space. The unscented transformation force fits the distribution of the residual energy to a Gaussian. The sum of the mean and deviation of the residual energy distribution was minimized which resulted in a robust input shaper. Clearly, this approach cannot represent non-Gaussian distributions which limits its capability.

This paper formulates an optimization problem exploiting the strengths of Polynomial Chaos to determine representative parameters (moments or cumulants) of the probability density function of states of a linear dynamical system whose model parameters are random variables. The Polynomial Chaos expansion provides a computationally efficient approach compared to Monte Carlo simulation for the estimation of moments or cumulants of the function of uncertain state variables. This paper is organized as follows: Following the introductory section, a brief overview of robust input shaper design is presented. This is followed by the development of the equations which represent the dynamics of the coefficients of the Polynomial Chaos series. The specific problems considered are a spring-mass system with an uncertain coefficient of stiffness and a three mass-spring system with uncertain mass and stiffness parameters. Numerical simulations are used to compare the results of robust minimax Input Shaper design with those of the Polynomial Chaos based design.

II. Input Shaping

Input-Shapers also referred to as time-delay filters are a simple and powerful approach for the shaping of reference input to eliminate or minimize residual motion of system undergoing transition from one set point to another. The system being controlled is assumed to be stable or marginally stable and could represent an open-loop or a closed-loop system. One of strengths of Input Shapers is their robustness to modeling uncertainties. The earliest solution to addressing the problem of sensitivity of the Input-Shaper to uncertainties in model parameter errors involved forcing the sensitivity of the residual energy to model errors to zero at the end of the maneuver. This works well for perturbations of the parameters about the nominal values. In case, one has knowledge of the domain of uncertainty and the distribution of the uncertain variable, this additional knowledge can be exploited in the design of a Input-Shaper by posing a minimax optimization problem. In this formulation, the maximum magnitude of the residual energy over the domain of uncertainty is minimized. The minimax optimization problem is computationally expensive and as the dimension of the uncertain space grows, the computational cost grows exponentially since the uncertain space
has to be finely sampled.

Let us consider a linear mechanical system of the following form:

\[
M(p)\ddot{x}(t, p) + C(p)\dot{x}(t, p) + K(p)x(t, p) = D(p)u(t)
\] (1)

where \(x(t, p) \in \mathbb{R}^n\) represents the generalized displacement coordinate with \(u(t) \in \mathbb{R}^m\) representing the deterministic system input vector. \(M, C(p)\) and \(K(p)\) are the mass, damping and stiffness matrices, respectively. \(p \in \mathbb{R}^r\) is a vector of uncertain system parameters which are functions of random variable \(\xi\) with known probability distribution function (pdf) \(f(\xi)\), i.e., \(p = p(\xi)\). For example, the uncertain variable \(p\) could be represented as:

\[
p \sim U[a, b]
\] (2)

which implies that \(p\) is a uniformly distributed random variable and lies in the range

\[
a \leq p \leq b.
\] (3)

or \(p\) could be a Gaussian variable

\[
p \sim \mathcal{N}[\mu, \Sigma]
\] (4)

where \(\mu\) is the mean of the random variable \(p\) and \(\Sigma\) is the covariance matrix. \(p\) is not restricted to uniform or Gaussian distributions. To demonstrate this, a polynomial based probability distribution function with a compact support will be used to illustrate the proposed design process.

The transfer function of the Input-shaper/Time-delay filter is parameterized as:

\[
G(s) = \sum_{i=0}^{Z} A_i e^{-sT_i}
\] (5)

where \(T_0\) is zero and \(Z\) is the number of delays in the pre-filter. \(A_i\) and \(T_i\) are the parameters that need to be determined to satisfy the objective of completing the state transition with minimal excursion from the desired final states in the presence of modeling uncertainties.

For rest-to-rest maneuvers, the residual energy is a germane cost function. The residual energy at the final time \(T_Z\) can be defined as:

\[
V(T_Z) = \frac{1}{2} \dot{x}^T(T_Z)M(p)\dot{x}(T_Z) + \frac{1}{2}(x(T_Z) - x_f)^T K(p)(x(T_Z) - x_f)
\] (6)

where \(x_f\) is the vector of the desired final displacement. The first term of \(V(T_Z)\) is the kinetic energy in the system and the last term is the pseudo-potential energy which measures the energy resident in a hypothetical
set of springs when $\mathbf{x}$ deviates from $\mathbf{x}_f$. If $\mathbf{K}(\mathbf{p})$ is not positive definite, the cost function $V(T_Z)$ has to be augmented with a quadratic terms to ensure that the cost function is positive definite. The resulting cost function is:

$$V(T_Z) = \frac{1}{2} \dot{\mathbf{x}}^T(T_Z) \mathbf{M}(\mathbf{p}) \dot{\mathbf{x}}(T_Z) + \frac{1}{2} (\mathbf{x}(T_Z) - \mathbf{x}_f)^T \mathbf{K}(\mathbf{p})(\mathbf{x}(T_Z) - \mathbf{x}_f) + \frac{1}{2} (x_r - x_{rf})^2 \quad (7)$$

where $x_r$ refers to the rigid body displacement and $x_{rf}$ is the desired final position.

The design of a robust Input-Shaper can be posed as the problem:

$$\min_{A_i, T_i} \max_{\mathbf{p}} V(T_Z) \quad (8)$$

which is a minimax optimization problem and emulates a Monte Carlo process. This as indicated earlier is a computationally expensive problem when the dimension of $\mathbf{p}$ grows. To alleviate this problem, this paper endeavors to identify a probabilistic representation of the uncertain residual energy $V(T_Z)$ as a function of the uncertain parameter vector $\mathbf{p}(\xi)$. To emulate the minimax optimization problem in the probability space, one would requires the distribution of $V(T_Z)$ to have a small mean and small variance which would correspond to small residual energy and small spread over the range of the random variable. One can add higher moments such as skew and kurtosis and minimizing the skew and kurtosis is consistent with the goal of minimizing the worst cost over the domain of uncertainty. In the next section, we describe a probabilistic method known as Polynomial Chaos which exploits knowledge of the domain and distribution of the uncertain model parameters to calculate approximately the moments or cumulents of the residual energy.

### III. Polynomial Chaos

Polynomial chaos is a term coined by Norbert Wiener in 1938, describing a method for representing an unknown stochastic variable as a probability distribution. The basic idea of this approach is to approximate the stochastic system state in terms of finite-dimensional series expansion in the stochastic space. The completeness of the space allows for the accurate representation of any PDF using a suitable basis. Certain bases can be chosen to represent given PDF with the fewest number of terms. For example, the Legendre polynomial can be used to represent the Uniform distribution with only two terms. For dynamical systems described by mathematical equations, the unknown coefficients are determined by minimizing an appropriate norm of the residual.

Let us consider a second order stochastic linear system given by Eq. \[1\]:

$$\mathbf{M}(\mathbf{p}) \ddot{\mathbf{x}}(t, \mathbf{p}) + \mathbf{C}(\mathbf{p}) \dot{\mathbf{x}}(t, \mathbf{p}) + \mathbf{K}(\mathbf{p}) \mathbf{x}(t, \mathbf{p}) = \mathbf{D}(\mathbf{p}) \mathbf{u}(t) \quad (9)$$
where $M \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times n}$, $K \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times m}$. As mentioned earlier, $p \in \mathbb{R}^r$ is a vector of uncertain system parameters which are functions of the random variable $\xi$ with known probability distribution function (pdf) $f(\xi)$, i.e., $p = p(\xi)$. It is assumed that the uncertain state vector $x(t, p)$ and system parameters, $M_{ij}$, $C_{ij}$ and $K_{ij}$ can be written as a linear combination of basis functions, $\phi_l(\xi)$, which span the stochastic space of random variable $\xi$.

\[
x_i(t, p) = \sum_{l=0}^{N} x_{li}(t)\phi_l(\xi) = x_i^T(t)\Phi(\xi)
\]

\[
M_{ij}(p) = \sum_{l=0}^{N} m_{ijl}\phi_l(\xi) = m_{ij}^T\Phi(\xi)
\]

\[
C_{ij}(p) = \sum_{l=0}^{N} c_{ijl}\phi_l(\xi) = c_{ij}^T\Phi(\xi)
\]

\[
K_{ij}(p) = \sum_{l=0}^{N} k_{ijl}\phi_l(\xi) = k_{ij}^T\Phi(\xi)
\]

\[
D_{ij}(p) = \sum_{l=0}^{N} d_{ijl}\phi_l(\xi) = d_{ij}^T\Phi(\xi)
\]

where $\Phi(.) \in \mathbb{R}^N$ is a vector of polynomials basis functions orthogonal to the pdf $f(\xi)$ which can be constructed using the Gram-Schmidt Orthogonalization Process. The coefficients $m_{ijl}$, $c_{ijl}$, $k_{ijl}$, and $d_{ijl}$ are obtained by making use of following normal equations:

\[
m_{ijl} = \frac{\langle M_{ij}(p(\xi)), \phi_l(\xi) \rangle}{\langle \phi_l(\xi), \phi_l(\xi) \rangle}
\]

\[
c_{ijl} = \frac{\langle C_{ij}(p(\xi)), \phi_l(\xi) \rangle}{\langle \phi_l(\xi), \phi_l(\xi) \rangle}
\]

\[
k_{ijl} = \frac{\langle K_{ij}(p(\xi)), \phi_l(\xi) \rangle}{\langle \phi_l(\xi), \phi_l(\xi) \rangle}
\]

\[
d_{ijl} = \frac{\langle D_{ij}(p(\xi)), \phi_l(\xi) \rangle}{\langle \phi_l(\xi), \phi_l(\xi) \rangle}
\]

where $\langle u(\xi), v(\xi) \rangle = \int_\Omega u(\xi)v(\xi)f(\xi)d\xi$ represents the norm introduced by pdf $f(\xi)$ with support $\Omega$. Now, substitution of Eq. (10), Eq. (11), Eq. (13) and Eq. (14) in Eq. (9) leads to

\[
e_i(\xi) = \sum_{j=1}^{n} \left( \sum_{l=0}^{N} m_{ijl}\phi_l(\xi) \right) \left( \sum_{l=0}^{N} \dddot{x}_{jl}(t)\phi_l(\xi) \right) + \sum_{j=1}^{n} \left( \sum_{l=0}^{N} c_{ijl}\phi_l(\xi) \right) \left( \sum_{l=0}^{N} \dddot{x}_{jl}(t)\phi_l(\xi) \right)
\]

\[
+ \sum_{j=1}^{n} \left( \sum_{l=0}^{N} k_{ijl}\phi_l(\xi) \right) \left( \sum_{l=0}^{N} x_{jl}(t)\phi_l(\xi) \right) - \sum_{j=1}^{m} \left( \sum_{l=0}^{N} d_{ijl}\phi_l(\xi) \right) u_j, \quad i = 1, 2, \ldots, n
\]

Now, $n(N+1)$ time-varying unknown coefficients $x_{i \cdot}(t)$ can be obtained by using the Galerkin discretization
process, i.e., projecting the error of Eq. (19) onto the space of basis functions $\phi_l(\xi)$.

$$\langle e_i(\xi), \phi_l(\xi) \rangle = 0, \quad i = 0, 1, 2, \ldots, n, \quad l = 1, 2, \ldots, N$$

(20)

where $e_i(\xi)$ is the approximation error resulting from representing the system states by a Polynomial Chaos expansion. This leads to following set of $n(N+1)$ deterministic differential equations:

$$M\ddot{x}_p(t) + C\dot{x}_p(t) + Kx_p(t) = Du(t)$$

(21)

where $x_p(t) = \{x_1^T(t), x_2^T(t), \ldots, x_n^T(t)\}^T$ is a vector of $n(N+1)$ unknown coefficients and $M \in \mathbb{R}^{n(N+1) \times n(N+1)}$, $C \in \mathbb{R}^{n(N+1) \times n(N+1)}$, $K \in \mathbb{R}^{n(N+1) \times n(N+1)}$ and $D \in \mathbb{R}^{n(N+1) \times m}$.

Let $P$ and $T_k$, for $k = 0, 1, 2, \ldots, N$, denote the inner product matrices of the orthogonal polynomials defined as follows:

$$P_{ij} = \langle \phi_i(\xi), \phi_j(\xi) \rangle, \quad i, j = 0, 1, 2, \ldots, N$$

(22)

$$T_{kj} = \langle \phi_i(\xi), \phi_j(\xi), \phi_k(\xi) \rangle, \quad i, j = 0, 1, 2, \ldots, N$$

(23)

Then $M$, $C$ and $K$ can be written as $n \times n$ matrix of block matrices, each block being an $(N+1) \times (N+1)$ matrix. The matrix $M$ consists of blocks $M_{ij} \in \mathbb{R}^{(N+1) \times (N+1)}$:

$$M_{ij} = M_{ij}P, \quad i, j = 1, 2, \ldots, n$$

(24)

if the mass matrix is not uncertain, else, it is given by:

$$M_{ij}(k,:) = m_{ij}^T T_k, \quad i, j = 1, 2, \ldots, n$$

(25)

Similarly, for the matrices $C$ and $K$, the $k^{th}$ row of each of their block matrices $C_{ij}, K_{ij} \in \mathbb{R}^{(N+1) \times (N+1)}$ is given by,

$$C_{ij}(k,:) = c_{ij}^T T_k, \quad i, j = 1, 2, \ldots, n$$

(26)

$$K_{ij}(k,:) = k_{ij}^T T_k, \quad i, j = 1, 2, \ldots, n$$

(27)

The matrix $D$ consists of blocks $D_{ij} \in \mathbb{R}^{(N+1) \times 1}$:

$$D_{ij} = Pd_{ij}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m$$

(28)
Eq. (10) along with Eq. (21) define the uncertain state vector $x(t, \xi)$ as a function of random variable $\xi$ and can be used to compute any order moment or cumulant of a function of uncertain state variable. For example, the first moment of residual energy can be computed as:

$$E[V(T_Z)] = \int_\Omega V(T_Z, \xi) f(\xi) d\xi = \frac{1}{2} x_p^T(T_Z)M x_p(T_Z) + \frac{1}{2} x_p^T(T_Z)K(p)x_p(T_Z) - \frac{1}{2} E[x_f^T x_f - 2x_f^T x_f]$$ (29)

The problem of the design of a robust Input-Shaper as described in Section II is one which minimizes the $\infty$ norm of residual energy over the domain of uncertain parameters. In a probabilistic framework, this corresponds to concurrently forcing the mean of the probability distribution of the residual energy to be a minimum and the distribution be as thin as possible. This can be achieved by posing a multi-objective optimization problem. The cost function is a weighted sum of the absolute values of the central moments. This cost function can be represented as:

$$\min_{A_i, T_i} \sum_{i=1}^P \alpha_i |E[(V(T_Z) - E[V(T_Z)])^i]|$$ (30)

where $\alpha_i > 0$ is a weighting parameter. Notice that $P = 2$ corresponds to minimization of mean and variance of the residual energy. In the next section, we illustrate the proposed procedure by considering two examples.

IV. Example (Single Spring-Mass System)

To illustrate the proposed technique of using Polynomial Chaos for the determination of a robust Input-shaper, we consider the second order system:

$$\ddot{x} + kx = ku$$ (31)

where $k$ is an uncertain parameter of the system which is known to lie between the interval $[a b]$. We assume it to be a function of random variable $\xi$ with known probability density function $f(\xi)$. Thus, the uncertain parameter $k$ can be represented as:

$$k(\xi) = \sum_{i=0}^N k_i \phi_i(\xi).$$ (32)

Furthermore, if $\xi \in [-1 1]$, only two terms are necessary to represent $k(\xi)$,

$$k(\xi) = k_0 + k_1 \xi, \quad k_0 = \frac{a + b}{2}, \quad k_1 = \frac{b - a}{2}.$$ (33)
This does not preclude Normal distributions, since $k_0$ and $k_1$ can represent the mean and variance of $k(\xi)$ and $\xi \in [-\infty, \infty]$.

Now, the displacement $x$ is represented as:

$$x = \sum_{i=0}^{N} x_i(t) \phi_i(\xi)$$  \hspace{1cm} (34)

where $\phi_i(\xi)$ represents the orthogonal polynomial set with respect to pdf $f(\xi)$, i.e.

$$\langle \phi_i(\xi), \phi_j(\xi) \rangle = \int_{\Omega} \phi_i(\xi)\phi_j(\xi)f(\xi)d\xi = c_i^2 \delta_{ij}$$  \hspace{1cm} (35)

For example, the Legendre and Hermite polynomials constitute the orthogonal polynomial sets for uniform and normal distribution, respectively. In general, these polynomials can be constructed by making use of the Gram-Schmidt Orthogonalization process. Now, substituting for $x$ and $k$ from Eqs. (34) and (32) in Eq. (31) leads to

$$\sum_{i=0}^{N} \phi_i(\xi)\ddot{x}_i + (k_0 \phi_0(\xi) + k_1 \phi_1(\xi)) \sum_{i=0}^{N} \phi_i(\xi)x_i = (k_0 \phi_0(\xi) + k_1 \phi_1(\xi)) u.$$  \hspace{1cm} (36)

Using the Galerkin projection method, the dynamics of $x_i$ can be determined. Making use of the fact that system equation error due to polynomial chaos approximation (Eq. (36)) should be orthogonal to basis function set $\phi_j(\xi)$, we arrive at the equation:

$$\mathbf{M} \begin{bmatrix} \ddot{x}_0 \\ \ddot{x}_1 \\ \vdots \\ \ddot{x}_N \end{bmatrix} + \mathbf{K} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \mathbf{Du}$$  \hspace{1cm} (37)

where the elements of the $\mathbf{M}$ matrix are

$$\mathbf{M}_{ij} = \langle \phi_i(\xi), \phi_j(\xi) \rangle = \int_{\Omega} \phi_i(\xi)\phi_j(\xi)f(\xi)d\xi = c_i^2 \delta_{ij} \text{ where } i, j = 0, 1, 2...N$$  \hspace{1cm} (38)

and the elements of the $\mathbf{K}$ matrix are given by

$$\mathbf{K}_{ij} = k_0 \langle \phi_i(\xi), \phi_j(\xi) \rangle + k_1 \langle \xi \phi_i(\xi), \phi_j(\xi) \rangle$$  \hspace{1cm} (39)
Now, making use of the fact that every orthogonal polynomial set satisfies a three-term recurrence relation:

\[ \xi \phi_n(\xi) = \frac{a_n}{a_{n+1}} \phi_{n+1}(\xi) + \frac{c_n^2}{c_{n-1}} \frac{a_{n-1}}{a_n} \phi_{n-1}(\xi) \]  

(40)

where \( a_n \) and \( a_{n-1} \) are the leading coefficients of \( \phi_n(\xi) \) and \( \phi_{n-1}(\xi) \), respectively. Now, making use of this recurrence relationship, the elements of the \( \mathcal{K} \) matrix are given by

\[ K_{ii} = k_0 \langle \phi_i(\xi), \phi_j(\xi) \rangle \]

(41)

\[ K_{i,i+1} = k_1 \langle \phi_{i+1}(\xi), \phi_j(\xi) \rangle = k_1 c_i^2 \frac{a_i}{a_{i+1}} \]

(42)

\[ K_{i,i-1} = k_1 c_i^2 \frac{a_{i-1}}{a_i} \]

(43)

and

\[ \mathcal{D} = \begin{bmatrix} c_0^2 k_0 \\ c_1^2 k_1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \]

(44)

Equation (37) can be easily solved for a parameterized \( u \) (Equation (5)). The residual energy at the final time \( T_Z \) can be represented as:

\[ V(T_Z, \xi) = \frac{1}{2} \left( \sum_{i=0}^{N} \dot{x}_i \phi_i(\xi) \right)^T \left( \sum_{i=0}^{N} \dot{x}_i \phi_i(\xi) \right) + \frac{1}{2} \left( \sum_{i=0}^{N} x_i \phi_i(\xi) - x_f \right)^T k(\xi) \left( \sum_{i=0}^{N} x_i \phi_i(\xi) - x_f \right) \]

(45)

The mean of the residual energy can be easily calculated using the equation

\[ \mathbb{E}[V(T_Z, \xi)] = \mu = \int_{\Omega} V(T_Z, \xi) f(\xi) \, d\xi \]

(46)

and the higher central moments by the equation:

\[ \mathbb{E}[(V(T_Z, \xi) - \mu)^n] = \int_{\Omega} (V(T_Z, \xi) - \mu)^n f(\xi) \, d\xi \]

(47)

A. Uniform Distribution

To illustrate the proposed technique of using Polynomial Chaos for the determination of a robust Input-shaper, consider the spring-mass system (Equation (31)) where \( k \) is a uniformly distributed random variable.
which lies in the range

\[
0.7 \leq k \leq 1.3.
\]  

(48)

The random variable \( k \) can be represented by Legendre polynomials \( (P_i) \) as:

\[
k(\xi) = k_0 P_0(\xi) + k_1 P_1(\xi)
\]  

(49)

where \( \xi \) is a uniformly distributed random variable which lies in the range \([-1, 1] \) and \( k_0 = \frac{a + b}{2} = 1 \) and \( k_1 = \frac{b - a}{2} = 0.3 \). Table 1 lists the basis functions and the coefficients which are necessary to construct the Polynomial Chaos coefficient dynamics which can be represented as:

\[
\begin{bmatrix}
\vec{x}_0 \\
\vec{x}_1 \\
\vdots \\
\vec{x}_{i-1} \\
\vec{x}_i \\
\vdots \\
\vec{x}_N
\end{bmatrix} =
\begin{bmatrix}
k_0 & \frac{1}{2} k_1 & 0 \\
k_1 & k_0 & \frac{2}{3} k_1 \\
\vdots & \vdots & \vdots \\
k_0 & \frac{i}{2i+1} k_1 & 0 \\
k_0 & \frac{(i+1)}{2i+3} k_1 \\
\vdots & \vdots & \vdots \\
0 & \frac{n}{2N-1} k_1 & k_0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{i-1} \\
x_i \\
\vdots \\
x_N
\end{bmatrix}
\]  

(50)

Equation (50) can be easily solved for a parameterized \( u \) (Equation 5). The residual energy at the final time \( T_Z \) can be represented as:

\[
V(T_Z, \xi) = \frac{1}{2} \left( \sum_{i=0}^{N} \dot{x}_i P_i(\xi) \right)^T \left( \sum_{i=0}^{N} \dot{x}_i P_i(\xi) \right) + \frac{1}{2} \left( \sum_{i=0}^{N} x_i P_i(\xi) - x_f \right)^T k(\xi) \left( \sum_{i=0}^{N} x_i P_i(\xi) - x_f \right)
\]  

(51)
The mean of the residual energy can be easily calculated using the equation

\[ E[V(T_Z, \xi)] = \mu = \frac{\int_{\Omega} V(T_Z, \xi) \frac{1}{2} d\xi}{\int_{\Omega} 1 \frac{1}{2} d\xi} \tag{52} \]

and the higher central moments by the equation:

\[ E[(V(T_Z, \xi) - \mu)^n] = \frac{\int_{\Omega} (V(T_Z, \xi) - \mu)^n \frac{1}{2} d\xi}{\int_{\Omega} 1 \frac{1}{2} d\xi} \tag{53} \]

Parameterize a two time-delay filter as:

\[ u = A_0 + A_1 H(t - T_1) + A_2 H(t - T_2) \tag{54} \]

with the constraint that \( A_0 + A_1 + A_2 = 1 \), to mandate the final value of the output of the time-delay filter subject to a step input will be the same as the magnitude of the input. The Polynomial Chaos expansion is used to represent the residual energy random variable \( V(T_2) \). A constrained minimization problem is solved to minimize a series of cost functions which are listed in Table 2. Figure 1 illustrates the performance of the Polynomial Chaos based design to the standard minimax controller design which endeavors to minimize the maximum magnitude of the residual energy over the domain of spring stiffness uncertainty. The distribution of the uncertain spring stiffness is assumed to be uniform. It is clear from Figure 1 that as higher moments are included in the design process, the resulting solutions tends towards the minimax solution. This is further evidenced in Figure 2 where a three time-delay Input Shaper design based on Polynomial Chaos and the minimax approach are compared.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Cost & \( A_0 \) & \( A_1 \) & \( A_2 \) & \( T_1 \) & \( T_2 \) \\
\hline
\( E[V(T_2)] = \mu \) & 0.2545 & 0.4909 & 0.2545 & 3.1416 & 6.2831 \\
\( \mu + \sqrt{E[(V(T_2) - \mu)^2]} \) & 0.2554 & 0.4892 & 0.2554 & 3.1416 & 6.2830 \\
\( \mu + \sqrt{E[(V(T_2) - \mu)^3]} + \sqrt[3]{E[(V(T_2) - \mu)^3]} \) & 0.2557 & 0.4886 & 0.2557 & 3.1415 & 6.2836 \\
\hline
\end{tabular}
\caption{Optimal Input Shaper (2 Delays)}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Cost & \( A_0 \) & \( A_1 \) & \( A_2 \) & \( A_3 \) & \( T_1 \) & \( T_2 \) & \( T_3 \) \\
\hline
\( E[V(T_2)] = \mu \) & 0.1810 & 0.4167 & 0.3223 & 0.0799 & 3.1365 & 6.2869 & 9.4237 \\
\( \mu + \sqrt{E[(V(T_2) - \mu)^2]} \) & 0.1810 & 0.4160 & 0.3231 & 0.0800 & 3.1380 & 6.2858 & 9.4240 \\
\( \mu + \sqrt{E[(V(T_2) - \mu)^3]} + \sqrt[3]{E[(V(T_2) - \mu)^3]} \) & 0.1357 & 0.3747 & 0.3658 & 0.1238 & 3.1500 & 6.2926 & 9.4190 \\
\hline
\end{tabular}
\caption{Optimal Input Shaper (3 Delays)}
\end{table}

To illustrate the ability of the Polynomial Chaos expansion to capture the distribution of the residual energy, the random coefficient of stiffness is sampled with 10,000 Monte Carlo samples and the distribution
Figure 1. PC Uniform distribution (2 delays filter)

Figure 2. PC Uniform distribution (3 delays filter)
of the residual energy is calculated. The mean and variance of the Monte Carlo based sampling is compared to those of the Polynomial Chaos based expansion. These results are illustrated in Figure 3 where the Monte Carlo based result is illustrated by the dashed line. The solid line is the estimate of the mean and variance as a function of the number of terms in the Polynomial Chaos expansion. It is clear that with a fourth order polynomial, the mean and variance have converged to the true values.

![Figure 3. Polynomial Chaos based Estimate of Mean and Variance](image)

B. Gaussian Distribution

One can determine a robust Input Shaper for normally distributed coefficient of stiffness similarly. Consider a random stiffness coefficient given by the equation

$$k = \mathcal{N}(1, 0.1^2)$$  \hspace{1cm} (55)

which corresponds to a Gaussian distribution with a mean of unity and a standard deviation of $\sigma = 0.1$. The variable $k$ can also be represented using Hermite polynomials as:

$$k = k_0 H_0(\xi) + k_1 H_1(\xi)$$  \hspace{1cm} (56)

where $k_0 = 1$, and $k_1 = 0.05$.

Assuming a Gaussian distributed random spring stiffness with a mean of unity and a deviation of 0.1, the residual energy represented by the Polynomial Chaos is determined and different combinations of the moments are minimized. The minimax problem is also solved to provide a benchmark to compare the
solution of the Polynomial Chaos expansion.

Figures 4 and 5 illustrate the results of the optimization problem which minimizes the mean, mean+deviation, mean+deviation+skew, and compares it to the minimax solution which is considered the desired solution. It is clear for both the two time-delay filter in Figure 4 and the three time-delay filter in Figure 5 that minimizes the mean results in a solution that is closest to the minimax solution.

<table>
<thead>
<tr>
<th>Index</th>
<th>Basis Functions ($H_i$)</th>
<th>$a_i$</th>
<th>$c_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1$</td>
<td>1</td>
<td>$\sqrt{2\pi}$</td>
</tr>
<tr>
<td>1</td>
<td>$\xi$</td>
<td>1</td>
<td>$4\sqrt{2\pi}$</td>
</tr>
<tr>
<td>2</td>
<td>$\xi^2 - 1$</td>
<td>1</td>
<td>$9\sqrt{2\pi}$</td>
</tr>
<tr>
<td>3</td>
<td>$\xi^3 - 3\xi$</td>
<td>1</td>
<td>$16\sqrt{2\pi}$</td>
</tr>
</tbody>
</table>

Table 4. Normal Distribution

<table>
<thead>
<tr>
<th>Cost</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$T_1$</th>
<th>$T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[V(T_2)] = \mu$</td>
<td>0.2515</td>
<td>0.4970</td>
<td>0.2515</td>
<td>3.1416</td>
<td>6.2831</td>
</tr>
<tr>
<td>$\mu + \sqrt{E[(V(T_2) - \mu)^2]}$</td>
<td>0.2527</td>
<td>0.4947</td>
<td>0.2526</td>
<td>3.1416</td>
<td>6.2830</td>
</tr>
<tr>
<td>$\mu + \sqrt{E[(V(T_2) - \mu)^3]} +</td>
<td>\sqrt{E[(V(T_2) - \mu)^3]}</td>
<td>$</td>
<td>0.2540</td>
<td>0.4923</td>
<td>0.2537</td>
</tr>
</tbody>
</table>

Table 5. Optimal Input Shaper (2 Delays)

Figure 4. PC Gaussian distribution (2 delays filter)
\[
E[V(T_2)] = \mu \\
\mu + \sqrt{E[(V(T_2) - \mu)^2]} \\
\mu + \sqrt{E[(V(T_2) - \mu)^2]} + |\sqrt{E[(V(T_2) - \mu)^3]}|
\]

<table>
<thead>
<tr>
<th>Cost</th>
<th>(A_0)</th>
<th>(A_1)</th>
<th>(A_2)</th>
<th>(A_3)</th>
<th>(T_1)</th>
<th>(T_2)</th>
<th>(T_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1276</td>
<td>0.3729</td>
<td>0.3725</td>
<td>0.1271</td>
<td>3.1433</td>
<td>6.2813</td>
<td>9.4145</td>
</tr>
<tr>
<td></td>
<td>0.1229</td>
<td>0.3694</td>
<td>0.3768</td>
<td>0.1309</td>
<td>3.1409</td>
<td>6.2792</td>
<td>9.4153</td>
</tr>
<tr>
<td></td>
<td>0.1179</td>
<td>0.3629</td>
<td>0.3817</td>
<td>0.1374</td>
<td>3.1421</td>
<td>6.2780</td>
<td>9.4061</td>
</tr>
</tbody>
</table>

Table 6. Optimal Input Shaper (3 Delays)

Figure 5. PC Gaussian distribution (3 delays filter)
C. Compact Support Polynomial Distribution

Singla and Junkins have developed polynomial functions with compact support as blending functions to permit \( w(t) \). These functions are:

\[
w(t) = 1 - K \sum_{n=0}^{m} A_n \xi^{2m-n+1},
\]

where \( K \) and \( A_n \) are given by the following expressions:

\[
K = \frac{(2m + 1)!(-1)^m}{(m!)^2}, \quad A_n = \frac{(-1)^n mC_n}{2m - n + 1}, \quad mC_n = \frac{m!}{n!(m-n)!}.
\]

Figure 6 illustrates five of these functions with the order of polynomial ranging from 0 which corresponds to the triangular shape to ten which tends to a top-hat function. This suite of functions are positive over the domain \( \xi \in [-1, 1] \) and have an enclosed area of one permitting them to be used as probability density functions.

For illustrative purposes, assume that the pdf of the stiffness is:

\[
f(\xi) = 1 - \xi^2 (3 - 2|\xi|).
\]

The Gram-Schmidt process can be used to determine the orthogonal basis functions \( \phi_i \). Assume that basis functions which we consider to construct the orthogonal polynomial bases are:

\[
\psi_i(\xi) = \xi^i \quad i = 0, 1, 2, \ldots
\]

Assume the first basis \( \phi_0 = \psi_0 = 1 \). The remaining basis \( \phi_i(\xi) \) can be constructed using the equation:

\[
\phi_i(\xi) = \psi_i(\xi) - \sum_{j=1}^{i-1} \frac{\langle \psi_i(\xi), \phi_j(\xi) \rangle}{\langle \phi_j(\xi), \phi_j(\xi) \rangle} \phi_j(\xi), \quad i = 1, 2, 3, \ldots
\]
Table 7 lists the orthonormal basis for the pdf given by Equation (59).

<table>
<thead>
<tr>
<th>Index</th>
<th>Basis Functions ( (G_i) )</th>
<th>( a_i )</th>
<th>( c_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \xi )</td>
<td>1</td>
<td>( \frac{2}{15} )</td>
</tr>
<tr>
<td>2</td>
<td>( \xi^2 - \frac{2}{15} )</td>
<td>1</td>
<td>( \frac{79}{3150} )</td>
</tr>
<tr>
<td>3</td>
<td>( \xi^3 - \frac{9}{28} \xi )</td>
<td>1</td>
<td>( \frac{31}{3850} )</td>
</tr>
</tbody>
</table>

Table 7. Compact Support Distribution

Assuming a spring stiffness distribution given by the equation:

\[
k(\xi) = k_0 G_0(\xi) + k_1 G_1(\xi) = 1 + 0.3\xi
\]  

(62)

and solving for the Input shaper which minimizes the mean of the distribution of the residual energy and its higher moments results in the shaper parameters presented in Tables 8 and 9.

Figures 7 and 8 illustrate the results of the optimization problem which minimizes the mean, mean+deviation, mean+deviation+skew, and compares it to the minimax solution which is considered the desired solution.

It is clear for both the two time-delay filter in Figure 4 and the three time-delay filter in Figure 5 that minimizes the mean results in a solution that is closest to the minimax solution.

<table>
<thead>
<tr>
<th>Cost</th>
<th>( A_0 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{E}[V(T_2)] = \mu )</td>
<td>0.2518</td>
<td>0.4964</td>
<td>0.2518</td>
<td>3.1358</td>
<td>6.27170</td>
</tr>
<tr>
<td>( \mu + \sqrt{\mathbb{E}[(V(T_2) - \mu)^2]} )</td>
<td>0.2528</td>
<td>0.4943</td>
<td>0.2528</td>
<td>3.1418</td>
<td>6.2835</td>
</tr>
<tr>
<td>( \mu + \sqrt{\mathbb{E}[(V(T_2) - \mu)^2]} +</td>
<td>\sqrt{\mathbb{E}[(V(T_2) - \mu)^3]}</td>
<td>)</td>
<td>0.2538</td>
<td>0.4923</td>
<td>0.2538</td>
</tr>
</tbody>
</table>

Table 8. Optimal Input Shaper (2 Delays)

<table>
<thead>
<tr>
<th>Cost</th>
<th>( A_0 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( T_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{E}[V(T_2)] = \mu )</td>
<td>0.1273</td>
<td>0.3728</td>
<td>0.3727</td>
<td>0.1273</td>
<td>3.1463</td>
<td>6.2928</td>
<td>9.4391</td>
</tr>
<tr>
<td>( \mu + \sqrt{\mathbb{E}[(V(T_2) - \mu)^2]} )</td>
<td>0.1282</td>
<td>0.3718</td>
<td>0.3718</td>
<td>0.1282</td>
<td>3.1532</td>
<td>6.3069</td>
<td>9.4602</td>
</tr>
<tr>
<td>( \mu + \sqrt{\mathbb{E}[(V(T_2) - \mu)^2]} +</td>
<td>\sqrt{\mathbb{E}[(V(T_2) - \mu)^3]}</td>
<td>)</td>
<td>0.1289</td>
<td>0.3711</td>
<td>0.3711</td>
<td>0.1289</td>
<td>3.1573</td>
</tr>
</tbody>
</table>

Table 9. Optimal Input Shaper (3 Delays)

V. Example (3-Mass-Spring System)

To illustrate the use of Polynomial Chaos expansion for solving robust Input shaper design problems for system with multiple uncertain parameters, we consider a three-mass spring system with two random variables. The three-mass spring system illustrated in Figure 9 can represent a simplified model of a double-pendulum crane. The variation in the cable length is represented as an uncertainty in the coefficient of
Figure 7. PC Compact Polynomial distribution (2 delays filter)

Figure 8. PC Compact Polynomial distribution (3 delays filter)
spring stiffness $k_1$. The variation in load carried by the crane is represented by assuming that the mass $m_3$ is uncertain. The system model is:

$$
\begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{bmatrix}
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3
\end{bmatrix}
+ \begin{bmatrix}
k_1 & -k_1 & 0 \\
-k_1 & k_1 + k_2 & -k_2 \\
0 & -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} u
$$

Assume that the constant system parameters are:

$$m_1 = 1, m_2 = 1, k_2 = 1,$$

and the uncertain parameters $k_1$ and $m_3$ are assumed to be represented as random variables:

$$k_1 \sim f_1(\xi_1), \text{ and } m_3 \sim f_2(\xi_2)$$

with probability density functions $f_1(\xi_1)$ and $f_2(\xi_2)$ respectively. The desire is to design a shaped reference profile which is robust to uncertainties in $k_1$ and $m_3$ and will be achieved by representing the system response to a parameterized reference input using Polynomial Chaos.

Since the system includes a rigid body mode, a PD controller

$$u = -k_p(y_1 - y_r) - k_d\dot{y}_1$$

is used to stabilize the system. The resulting closed loop system is:

$$
\begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{bmatrix}
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3
\end{bmatrix}
+ \begin{bmatrix}
k_1 + k_p & -k_1 & 0 \\
-k_1 & k_1 + k_2 & -k_2 \\
0 & -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
= \begin{bmatrix}
k_p \\
0 \\
0
\end{bmatrix} y_r
$$
The reference input \( y_r \) is shaped by filtering a unit step input through a time-delay filter parameterized using \( A_i \) and \( T_i \) as:

\[
Y_R(s) = \frac{1}{s} \sum_{i=0}^{Z} A_i \exp(-sT_i)
\]  

(68)

where \( T_0=0 \). A robust filter which minimizes the worst performance of the system over the domain of uncertainties in \( m_3 \) and \( k_1 \) is posed as an optimization problem when the expected mean and higher central moments of the distribution of the residual energy is minimized. To ensure that the final value of the shaped profile is the same as the reference input, we require:

\[
\sum_{i=0}^{Z} A_i = 1.
\]  

(69)

The uncertain model parameters are represented as a vector \( \mathbf{p} \) of the uncorrelated random variables \( k_1 \) and \( m_3 \) of known pdfs \( f(\xi) \), where \( \xi \) is a two dimensional vector. The random variable \( \mathbf{p} \) can be expanded using a Polynomial Chaos, permitting the states and the elements of the mass, damping, stiffness and control influence matrices to be represented using Equations 10-14. The orthogonal polynomials used in the Polynomial Chaos expansion are constructed by the tensor product of 1-D polynomials in the \( \xi_1, \xi_2 \) space assuming the joint pdf of the random variable is given by the equation:

\[
f(\xi) = f_1(\xi_1)f_2(\xi_2).
\]  

(70)

The resulting orthogonal polynomials are given as:

\[
\phi_{ij}(\xi_1, \xi_2) = \phi_i(\xi_1)\phi_j(\xi_2), \quad \forall i = 0, 1, \ldots, N, \quad j = 0, 1, \ldots, N, \quad \text{where } i + j \leq N.
\]  

(71)

Figure 11 illustrates the first ten 2-dimensional orthogonal basis functions for the probability density function \( f(\xi_1, \xi_2) = (1 - \xi_1^2(3 - 2\xi_1))(1 - \xi_2^2(3 - 2\xi_2)) \).

The dynamic model for the coefficients of the polynomials expansion used to represent the system states are derived using the Galerkin projection method described in Section III exploiting the property that:

\[
\langle \phi_{ij}(\xi_1, \xi_2), \phi_{lm}(\xi_1, \xi_2) \rangle = \int_{\Omega_1} \int_{\Omega_2} f(\xi_1)f(\xi_2)\phi_{ij}(\xi_1, \xi_2)\phi_{lm}(\xi_1, \xi_2) \, d\xi_1 d\xi_2
\]  

(72)

\[
= \int_{\Omega_1} f(\xi_1)\phi_i(\xi_1) \, d\xi_1 \int_{\Omega_2} f(\xi_2)\phi_j(\xi_2) \, d\xi_2
\]  

(73)

\[
= \epsilon_i^2 \epsilon_j^2 \delta(i-l)\delta(j-m).
\]  

(74)

This process is illustrated assuming that the uncertain parameters \( k_1 \) and \( m_3 \) can be represented either by
a Gaussian distribution:

\[ k_1 \sim \mathcal{N}[1, 0.04], \quad \text{and} \quad m_3 \sim \mathcal{N}[2, 0.16] \]  

(75)

or a uniform distribution:

\[ k_1 \sim \mathcal{U}[0.7, 1.3], \quad \text{and} \quad m_3 \sim \mathcal{U}[1.4, 2.6]. \]  

(76)

A. Uniform Distribution

Representing the uncertain parameters for a uniformly distributed density function as:

\[ k_1 = k_0^1 P_0(\xi_1) + k_1^1 P_1(\xi_1) \]  

(77)

and

\[ m_3 = m_0^3 P_0(\xi_2) + m_3^3 P_1(\xi_2) \]  

(78)

where \( P_i \) are Legendre polynomials, and \( \xi_1 \) and \( \xi_2 \) lie in the range \([-1, 1] \), we have \( k_0^1 = \frac{a+b}{2} = 1 \) and \( k_1^1 = \frac{a-b}{2} = 0.3 \) and \( m_0^3 = 2 \) and \( m_3^3 = 0.6 \).

Assuming \( N = 2 \), the system states can be represented in terms of Legendre polynomials which are functions of the two random variables as:

\[ x_i(t, \xi_1, \xi_2) = x_i^0(t) + x_i^1(t)\xi_1 + x_i^2(t)\xi_2 + x_i^3(t)\frac{1}{2}(3\xi_1^2 - 1) + x_i^4(t)\frac{1}{2}(3\xi_2^2 - 1) + x_i^5(t)\xi_1\xi_2 \]  

(79)
The Galerkin projection leads to the equation:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{D}_{x_r}$$  \hspace{1cm} (80)

where $\mathbf{X} \in \mathbb{R}^{n(N+1)(N+2)/2}$. $n$ and $N$ refer to the number of states and order of polynomials used in the expansion. The shaped reference input is parameterized using a seven time-delay filter and three cost functions which are functions of central moments of the residual energy are minimized. A minimax optimization problem is solved where the largest magnitude of the residual energy over the domain of uncertain parameters $(k_1, m_3)$ is minimized to permit comparison with the performance of the Polynomial Chaos based robust design. Figure 11 illustrates the variation of the residual energy over the uncertain domain. Figure 11 corresponds to the minimax solution and is used as the benchmark to compare the other three graphs. Figure 11 corresponds to the solution resulting from the minimization of the expected value of the residual energy, Figure 11: corresponds to the solution resulting from the minimization of the sum of the expected value of the residual energy and variance of the residual energy. The final plot, Figure 11: illustrates that minimizing the sum of the expected value of the residual energy and variance of the residual energy and the absolute value of the skew generates a time-delay filter whose performance is comparable to that resulting from the minimax solution.

B. Gaussian Distribution

Representing the uncertain parameters for a Normally distributed density function as:

$$k_1 = k_1^0H_0(\xi_1) + k_1^1H_1(\xi_1)$$ \hspace{1cm} (81)

and

$$m_3 = m_3^0H_0(\xi_2) + m_3^1H_1(\xi_2)$$ \hspace{1cm} (82)

where $H_i$ are Hermite polynomials, we have $k_1^0 = 1$ and $k_1^1 = 0.2$ and $m_3^0 = 2$ and $m_3^1 = 0.4$, we can use Equation 79 to represent the system states by replacing the Legendre polynomials $P_i$ with Hermite polynomials. The Polynomial Chaos expansion can be used to represent the residual energy permitting the calculation of the the expected value of the mean, mean+deviation and mean+deviation+absolute value of skew etc., whose minimization results in the variation of the residual energy illustrated in Figure 12a, 12b and 12c respectively. Figure 12a is the minimax solution which permits comparing the Polynomial Chaos based solution.
VI. Conclusions

This paper exploits the Polynomial Chaos approximation to represent the residual energy random variable. A general development which permits the use of any probability density function in conjunction with a polynomial chaos based representation of the uncertain variables and the systems states is presented. Three probability density function are considered: the first where the uncertain variable is uniformly distributed and the second where the distribution is Gaussian and the third which is a polynomial function with a compact support. Legendre and Hermite polynomials are used in the Polynomial Chaos approximation for the Uniform and Gaussian cases respectively. For the polynomial function with a compact support, the Gram-Schmidt process is used to generate orthogonal functions which are used in the Polynomial Chaos expansion. Simulation results for a single spring-mass system illustrate that minimizing the mean and combinations of the higher moments can result in an Input Shaper which closely approximates the minimax solution. The benefit of using Polynomial Chaos approximation to represent the random cost function compared to
the minimax approach is the reduction in the computational cost, since one does not need to sample the multi-dimensional uncertain space which is required for the minimax approach. A three mass-spring system is used to illustrate how the proposed approach can be easily extended to systems with multiple uncorrelated uncertainties. Minimizing the expected value of the mean, variance and absolute value of the skew is shown to progressively approach the solution of the minimax design.

References


