Uncertainty Evolution In Stochastic Dynamic Models Using Polynomial Chaos

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Abstract We present a new approach to describe the evolution of uncertainty in linear dynamic models with parametric and initial condition uncertainties, and driven by additive white Gaussian forcing. This is based on the polynomial chaos (PC) series expansion of second order random processes, which has been used in several domains to solve stochastic systems with parametric and initial condition uncertainties. The PC solution is found to be an accurate approximation to ground truth, established by Monte Carlo simulation, while offering an efficient computational approach for large nonlinear systems with a relatively small number of uncertainties. However, when the dynamic system includes an additive stochastic forcing term varying with time, using PC series expansion for the stochastic forcing term is computationally expensive and increase exponentially with the increase in the number of time steps, due to the increase in the stochastic dimensionality. For a model driven by white noise, which is uncorrelated in time, an infinite number of terms is required to model the white noise process. In this work, alternative approaches are proposed for uncertainty evolution in linear uncertain models driven by white noise. The uncertainty in the model states...
due to additive white Gaussian noise can be described by the mean and covariance of the states. This is combined with the PC based approach to propagate the uncertainty due to Gaussian stochastic forcing and model parameter uncertainties which can be non-Gaussian.

**Keywords** Polynomial chaos · uncertainty evolution · stochastic dynamic models · non-Gaussian distributions · stochastic forcing

1 Introduction

The mathematic models used to represent dynamical systems and physical processes are often incomplete and stochastic in nature. The solution $x$ of these models is therefore uncertain and is described by a time-dependent probability density function (pdf) $\pi(t, x)$. The uncertainty in these models is either due to a lack of complete description of the system or due to the inherent stochastic nature of the process. Such models are characterized by uncertain model parameters and stochastic forcing terms. The uncertainty in the solution of the models could also arise due to the uncertainty in initial and boundary conditions driving the models. The knowledge of the evolution of these uncertainties through the model is important to accurately quantify the uncertainty in the solution of the model at any future time.

Uncertainty propagation in various kinds of dynamical systems and physical processes has been studied extensively in various fields of engineering, finance, physical and environmental sciences [1, 19, 18, 12, 15]. Most of the methods incorporate linear approximations to nonlinear system response, or involve propagating only a few moments (often, just the mean and the covariance) of the distribution. Adjoint models [5] and parametric differentiation belong to this class and are widely used in sensitivity analysis of the models. These work well if the amount of uncertainty is small and there is adequate local linearity. Another class of methods, often used with models involving nonlinearities, are the various sampling strategies [8]. The uncertainty distributions are taken into account by appropriately sampling values from known or approximated prior distributions and the model is run repeatedly for those values to obtain a distribution of the outputs. A Gaussian mixture based approach is proposed in [16] for accurate uncertainty propagation through nonlinear dynamical systems due to uncertain initial conditions.

Polynomial chaos is a term originated by Norbert Wiener in 1938 [17], to describe the members of the span of Hermite polynomial functionals of a Gaussian process. According to the Cameron-Martin Theorem[3], the Fourier-Hermite polynomial chaos expansion converges, in the $L^2$ sense, to any arbitrary process with finite variance (which applies to most physical processes). This approach is combined with the finite element method to model uncertainty in [7]. This has been generalized in [20] to efficiently use the orthogonal polynomials from the Askey-scheme to model various probability distributions. This approach has been applied in modeling parametric uncertainties in multibody dynamical systems[14], structural mechanics [6], and computational fluid dynamics [11, 10]. This approach has been used in nonlinear system analysis for assessing stability and controller design [9].

While polynomial chaos presents an efficient alternative to Monte Carlo simulation in many cases, it becomes computationally expensive as the number of random parameters increases [2]. When the dynamic system includes an additive stochastic forcing
term varying with time, using polynomial chaos series expansion for the stochastic forcing term is computationally expensive. For stochastic processes which are correlated in time, a smaller set of terms in the expansion is enough to model the random processes to keep the computation feasible [7, 14]. For a model driven by white noise, which is uncorrelated in time, an infinite number of terms is required to model the white noise process. The computational costs increase exponentially with the increase in the number of time steps, due to the increase in the stochastic dimensionality. In this work, alternative approaches are proposed for uncertainty evolution in models driven by white noise. These methods are proposed for linear dynamic models where the solution is analytically shown to be accurate (?).

The present work proposes two approaches to efficiently approximate the solution of a linear uncertain dynamical model driven by additive white Gaussian stochastic forcing terms and uncertain initial conditions. The pdf of the uncertain model parameters is assumed to be known and can be non-Gaussian. The evolution of uncertainty in a deterministic model can be described using the state mean and covariance propagation when the forcing term is additive white Gaussian noise. This approach is combined with polynomial chaos to describe the evolution of a combination of uncertainties in model parameters, initial conditions and forcing terms.

The polynomial chaos approach to describe the evolution of uncertainties in model parameters is described in Section 2. This approach is extended to describe the evolution of uncertainty due to Gaussian stochastic forcing in Section 3. These methods are discussed using some examples in Section 4. The results are numerically compared with the Monte Carlo solutions for the examples. The conclusions and directions for future work are presented in Section 5.

2 Polynomial Chaos

The polynomial chaos theory is applied for the solution of a system of ordinary difference equations (ODEs). A dynamical system with uncertainties represented by a set of ODEs with stochastic parameters, can be transformed into a deterministic system of equations in the coefficients of a series expansion using this approach. The development in this section largely follows [7] and [20]. The basic goal of the approach is to approximate the stochastic system states in terms of a finite-dimensional series expansion in the infinite-dimensional stochastic space. The completeness of the space allows for the accurate representation of any random variable, with a given probability density function (pdf), by a suitable projection on the space spanned by a carefully selected basis. The basis can be chosen for a given pdf, to represent the random variable with the fewest number of terms. For example, the Hermite polynomial basis can be used to represent random variables with Gaussian distribution using only two terms. For dynamical systems described by parameterized models, the unknown coefficients are determined by minimizing an appropriate norm of the residual.

Let us consider a dynamical system of the form:

$$x(k + 1; p) = A(p)x(k; p) + B(p)u(k)$$

where, $x(k; p) \in \mathbb{R}^n$ represents the stochastic system state vector, $u(k) \in \mathbb{R}^n$ represents the deterministic input at time step $k$ and $p \in \mathbb{R}^m$ represents the vector of uncertain system parameters, which is a function of a random vector $\xi$, with components $\xi_j$ having a known pdf $g(\xi_j)$, with common support $\Omega$. Now, each of the uncertain states
and parameters can be expanded approximately by the finite dimensional Wiener-Askey polynomial chaos [20] as:

\[ x_i(k, p) = \sum_{r=0}^{P} x_{ir}(k) \phi_r(\xi) = x_i^T(k) \Phi(\xi) \]  

(2)

\[ p_j(\xi) = \sum_{r=0}^{P} p_{jr} \phi_r(\xi) = p_j^T \Phi(\xi) \]  

(3)

Using Eq. (3), the elements of \( A \) and \( B \) can be expanded as follows:

\[ A_{ij}(p) = \sum_{r=0}^{P} a_{ijr} \phi_r(\xi) = a_{ij}^T \Phi(\xi) \]  

(4)

\[ B_{ij}(p) = \sum_{r=0}^{P} b_{ijr} \phi_r(\xi) = b_{ij}^T \Phi(\xi) \]  

(5)

The coefficients in the PC expansion are evaluated using,

\[ a_{ijr} = \frac{\langle A_{ij}(p(\xi)), \phi_r(\xi) \rangle}{\langle \phi_r(\xi), \phi_r(\xi) \rangle} \]

\[ b_{ijr} = \frac{\langle B_{ij}(p(\xi)), \phi_r(\xi) \rangle}{\langle \phi_r(\xi), \phi_r(\xi) \rangle} \]

where \( \langle u(\xi), v(\xi) \rangle = \int_{\Omega} u(\xi)v(\xi)g(\xi)d\xi \) represents the inner product induced by pdf \( g(\xi) \). For linear and polynomial functions, the integrals in the inner products can be easily evaluated analytically [7] to obtain the coefficients. For non-polynomial nonlinearities, these integrals represent a challenge. Numerical quadrature methods are used to evaluate the multi-dimensional integrals in the present work. For instance, Gauss-Hermite quadrature formulae may be used to evaluate the integrals for a Hermite polynomial basis. These quadrature methods fall under the broader sampling-based Non-Intrusive Spectral Projection (NISP) methods discussed in [13]. The total number of terms in the polynomial chaos (PC) expansion is \( P + 1 \) and is determined by the chosen highest order \( l \) of the polynomials \( \{ \phi_r \} \) and the dimension \( m \) of the vector of uncertain parameters \( p \):

\[ P + 1 = \frac{(l + m)!}{l!m!} \]  

(6)

2.1 Example

For a dynamical system with two independent uncertain parameters, each of which is a Gaussian random variable \( \mathcal{N}(\mu_j, \sigma_j^2) \) for \( j = 1, 2 \), Hermite polynomials are chosen as the random orthogonal basis functions for expansion. Each of the uncertain parameters is expressed as a function of an independent Gaussian random variable \( \xi_j \in \mathcal{N}(0, 1) \). The orthogonal polynomials used in the polynomial chaos expansion are constructed as the tensor products of 1-D Hermite polynomials in the \( \xi_1, \xi_2 \) space, the joint pdf of the random variables being given by:

\[ g(\xi) = g(\xi_1)g(\xi_2) \]

(7)

The resulting orthogonal polynomials up to order \( l \) are given by:

\[ \phi(\xi_1, \xi_2) = \phi_i(\xi_1)\phi_j(\xi_2), \]

\[ \forall i = 0, 1, \ldots, l, \ j = 0, 1, \ldots, l, \ \text{where} \ i + j \leq l. \]
The Hermite polynomials up to, say, order \( l = 2 \) can be written as:

\[
\begin{align*}
\phi_0(\xi_1, \xi_2) &= 1 \\
\phi_1(\xi_1, \xi_2) &= \xi_1 \\
\phi_2(\xi_1, \xi_2) &= \xi_2 \\
\phi_3(\xi_1, \xi_2) &= \xi_1^2 - 1 \\
\phi_4(\xi_1, \xi_2) &= \xi_1 \xi_2 \\
\phi_5(\xi_1, \xi_2) &= \xi_2^2 - 1
\end{align*}
\]

where each \( \xi_i \) is \( \mathcal{N}(0, 1) \). Note that there are six terms in the expansion as given by Eq. (6). The system states can be written as:

\[
x_i(k; \xi_1, \xi_2) = \sum_{r=0}^{5} x_{ir}(k) \phi_r(\xi_1, \xi_2), \text{ for } i = 1, \ldots, n
\]

(8)

The two uncertain parameters can be expanded as:

\[
p_j(\xi_1, \xi_2) = \mu_j + \sigma_j \xi_j, \text{ for } j = 1, 2
\]

(9)

Note that the coefficients for the other terms are all zero.

2.2 Solution

Substitution of the approximate expressions for \( x \) and \( p \) in Eq. (2) and Eq. (3), in Eq. (1) leads to:

\[
e_i(\xi) = x_i^T(k+1)\Phi(\xi) - \sum_{j=1}^{n} a_{ij}^T \Phi(\xi)x_j^T(k)\Phi(\xi) - \sum_{j=1}^{n} b_{ij}^T \Phi(\xi)u_j
\]

for \( i = 1, \ldots, n \)

(10)

where, \( e(\xi) \) represents the error due to the truncated polynomial chaos expansions of \( x \) and \( p \). The \( n(P+1) \) time-varying unknown coefficients \( x_{ir} \) can be obtained using the Galerkin projection method. Projecting the error onto the space of basis functions \( \{\phi_r\} \) and minimizing it leads to \( n(P+1) \) deterministic equations:

\[
(x_{ir}(k+1), \phi_r^2) - \sum_{j=1}^{n} a_{ij}^T \Phi(\xi)x_j^T(k)\Phi(\xi) - \sum_{j=1}^{n} b_{ij}^T \Phi(\xi)u_j(\xi), \phi_r = 0
\]

(11)

These integrals can be evaluated analytically for linear systems to get a set of deterministic ODEs with the coefficients of the polynomial chaos series expansions as the states:

\[
c(k+1) = A_p c(k) + B_p u_p(k)
\]

(12)

where, \( c(k) = [x_1^T(k), x_2^T(k), \ldots, x_n^T(k)]^T \in \mathbb{R}^{n(P+1)} \) is a vector of the PC coefficients, \( A_p \) is the deterministic system matrix and \( B_p \) is the deterministic input matrix corresponding to the transformed input vector \( u_p(k) \). These equations can then be solved
to obtain the time-history of the time-varying coefficients \( x_{ir} \). The solution of the stochastic system in Eq. (1) can thus be obtained in terms of polynomial functionals of random variables \( \xi \):

\[
x_{i}(k, p) = \sum_{r=0}^{P} x_{ir}(k) \phi_{r}(\xi), \quad i = 1, \ldots, n
\]

This expression can be used to estimate the pdf of the solution by Monte Carlo sampling of the random vector \( \xi \). Further, the first coefficient \( x_{i0}(k) \) represents the mean of the solution \( x_{i}(k, p) \) when \( \phi_{0}(\xi) = 1 \). The other coefficients similarly represent the combinations of various moments of the solution:

\[
x_{ir}(k) = \left< x_{i}(k, p) \phi_{r}(\xi) \right>
\]

The mean of the solution \( x_{i}(k, \xi) \) can be evaluated analytically using the equation

\[
E[x_{i}(k, p(\xi))] = \mu(k) = \int_{\Omega} x_{i}(k, p(\xi)) g(\xi) d\xi
\]

and the \( n^{th} \) central moment by the equation:

\[
E[(x_{i}(k, p(\xi)) - \mu(k))^n] = \int_{\Omega} (x_{i}(k, p(\xi)) - \mu(k))^n g(\xi) d\xi
\]

For orthogonal basis functions \( \{ \phi_{r} \} \) with \( \phi_{0} = 1 \), it can be verified from these equations that

Mean, \( \mu(k) = E[x_{i}(k, p(\xi))] = x_{i0}(k) \)

Variance, \( \sigma^{2}(k) = E[(x_{i}(k, p(\xi)) - \mu(k))^{2}] = \sum_{r=1}^{P} x_{ir}^{2}(k) \left< \phi_{r}^{2} \right> \)

The pdf of the solution can also be estimated from the evaluated finite number of moments of the solution by maximizing the entropy of the solution[4].

3 Uncertainty evolution under stochastic forcing

The system described by Eq. (1) is now considered with an additive white Gaussian noise (AWGN) as the stochastic forcing term \( w \), in addition to the uncertain model parameters \( p \):

\[
x(k+1; p) = A(p)x(k; p) + B(p)u(k) + w(k)
\]

where, \( x(k; p, \xi, \omega) \in \mathbb{R}^{n} \) represents the stochastic system state vector at time step \( k \), and \( w(k) \in \mathbb{R}^{n} \) is a zero mean white Gaussian noise vector \( s(\omega) = \mathcal{N}(0, Q(k)) \), \( Q(k) \) being the covariance matrix of the noise vector \( w(\omega) \) at time step \( k \). This white noise process is uncorrelated in time and with other uncertainties in model parameters and initial conditions. Since the stochastic forcing terms are independent random variables at different times, using polynomial chaos series expansion to model the stochastic
forcing terms is computationally expensive. The computational costs increase exponentially with the increase in the number of time steps. Two approaches are proposed in this work for the accurate description of uncertainty evolution in dynamic models with parametric uncertainty and driven by AWGN. These approaches, briefly illustrated in Fig. 1, are discussed in the following.

![Diagram of Uncertainty Evolution](image)

**Fig. 1** Uncertainty evolution under stochastic forcing

### 3.1 Propagation of the first two moments of the PC coefficients of the states

Following Eq. (2), each of the uncertain states can be expanded approximately by the PC series expansion as:

\[
x_i(k;p(\xi),\omega) = \sum_{r=0}^{P} x_{ir}(k;\omega)\phi_r(\xi) = x_i^T(k;\omega)\Phi(\xi)
\]  

In such an expansion, the coefficients \(x_{ir}\) of the PC series expansion of the states are not just deterministic functions of time but random processes due to the stochastic forcing terms. Substitution of the approximate expressions for \(x\) and \(p\) in Eq. (18) and Eq. (3), in Eq. (17) leads to:

\[
e_i(\xi) = x_i^T(k+1)\Phi(\xi) = \sum_{j=1}^{n} a_{ij}^T\Phi(\xi)x_j^T(k)\Phi(\xi) - \sum_{j=1}^{n} b_{ij}^T\Phi(\xi)u_j(k) - w_i(k) \
\]  

for \(i = 1, \ldots, n\)  

where, \(e_i(\xi)\) represents the error due to the truncated polynomial chaos expansions of \(x\) and \(p\). The \(n(P+1)\) time-varying and unknown coefficients \(x_{ir}\) can be obtained using the Galerkin projection method, as described in the previous section. Projecting the error onto the space of orthogonal basis functions \(\{\phi_r\}\) and minimizing it leads to
The complete distribution of the original states is propagated using the state mean and covariance propagation equations for the model: linear, the states series expansion of the uncertain parameters and states. Since the model in Eq. (22) is a model with deterministic parameters driven by AWGN in Eq. (22), using the PC uncertain parameters and AWGN forcing terms described in Eq. (17), is reduced to a model with deterministic parameters driven by AWGN in Eq. (22), using the PC series expansion of the uncertain parameters and states. Since the model in Eq. (22) is linear, the states are also Gaussian. The two moments of these states can then be propagated using the state mean and covariance propagation equations for the model:

\[ \tilde{c}(k+1) = A_p \tilde{c}(k) + B_p u_p(k) \]

\[ P_c(k+1) = A_p P_c(k) A_p^T + Q_p(k) \]

where, \( \tilde{c} \) is the mean and \( P_c \) is the covariance matrix of the vector of PC coefficients \( c \). Given the mean and covariance of the Gaussian random vector \( c(k) \), different realizations of the PC coefficients \( x_i(k, \omega) \) can be obtained. Each such realization of \( c \) represents a distribution of the uncertainty in states \( x \) corresponding to a particular white noise sequence. The complete distribution of the original states \( x_i \) at any time step \( k \), can then be estimated from Eq. (18), using independent Monte Carlo sampling of \( c \) and \( \xi \).

In this approach, the pdf \( \pi \) of the uncertain states \( x_i(k, \xi, \omega) \) is obtained from the following:

\[ \pi(x_i(k, \xi, \omega)) = \int_{-\infty}^{\infty} \pi(x_i(k, \xi, \omega) | x_i(\omega)) \pi(x_i(\omega)) d\omega \]

\[ = \int_{-\infty}^{\infty} \pi(x_i(k, \xi, \omega) | x_i(\omega)) s(x_i(\omega)) d\omega \]
The pdf $\pi$ of the uncertain states $x_i(k, \xi, \omega)$ can then be estimated from their moments, which can be evaluated analytically. The mean of the solution $x_i(k, \xi, \omega)$ can be evaluated using the equation:

$$
E[x_i(k, \xi, \omega)] = \mu(k) = \int_{-\infty}^{\infty} E[x_i(k, \xi, \omega)|x_i(\omega)] s(x_i(\omega)) d\omega
$$

$$
= \int_{-\infty}^{\infty} \left( \int_{\Omega} x_i(k, \xi, \omega) \pi(x_i|\xi, \omega) dx_i \right) s(x_i(\omega)) d\omega \quad (25)
$$

For orthogonal basis functions $\{\phi_r\}$ with $\phi_0 = 1$, it can be verified that

$$
\mu(k) = \int_{-\infty}^{\infty} \left( \int_{\Omega} \left( \sum_{r=0}^{P} x_{i_r}(k; \omega) \phi_r(\xi) \right) g(\xi) d\xi \right) s(x_i(\omega)) d\omega
$$

$$
= \int_{-\infty}^{\infty} x_{i_0}(k, \omega) s(x_{i_0}(\omega)) d\omega
$$

$$
= \hat{x}_{i_0}(k) \quad (26)
$$

Similarly, the $n^{th}$ central moment can be evaluated using the equation:

$$
E[(x_i(k, \xi, \omega) - \mu(k))^n] = \int_{-\infty}^{\infty} E[(x_i(k, \xi, \omega) - \mu(k))^n|x_i(\omega)] s(x_i(\omega)) d\omega
$$

$$
= \int_{-\infty}^{\infty} \left( \int_{\Omega} (x_i(k, \xi, \omega) - \mu(k))^n \pi(x_i|\xi, \omega) dx_i \right) s(x_i(\omega)) d\omega
$$

$$
= \int_{-\infty}^{\infty} x_{i_0}(k, \omega) s(x_{i_0}(\omega)) d\omega \quad (27)
$$

From these, the pdf of the solution can be obtained by maximizing the entropy given a finite number of moments.

### 3.2 Propagation of the PC coefficients of the first two moments of the states

For a particular realization of the uncertain model parameters, the system states of the linear model described by Eq. (17) are Gaussian due to AWGN. The two moments of the model states $x$ are given by the state mean and covariance propagation equations for the model:

$$
\dot{x}(k+1) = A(p)\hat{x}(k) + B(p)u(k)
$$

$$
P_x(k+1) = A(p)P_x(k)A^T(p) + Q(k) \quad (28)
$$

where, $\hat{x}$ is the mean and $P_x$ is the covariance matrix of the system state vector $x$. These propagation equations are deterministic and depend only on the initial conditions (of mean and covariance terms) and the model parameters. Combining the state mean and covariance terms into a new state vector $z \in \mathbb{R}^N$ where $N = n(n + 3)/2$, the propagation equations represent an augmented model describing the evolution of $z$ without a stochastic forcing term:

$$
z(k+1) = A_w(p)z(k) + B_w(p)u_w(k) \quad (29)
$$

where, $A_w(p)$ is the augmented system matrix, $B_w(p)$ is the augmented input matrix corresponding to the transformed input vector $u_w(k)$ for the augmented model. This
augmented model, which is similar in form to Eq. (1), can be used to estimate the effect of uncertain model parameters $p$ on the evolution of $z$, using polynomial chaos as described in section 2. The solution of the stochastic system in Eq. (29) can thus be obtained in terms of polynomial functionals of random variables $\xi_i$:

$$z_i(k, p) = \sum_{r=0}^{P} z_{ir}(k) \phi_r(\xi), \quad i = 1, \ldots, N$$  \hspace{1cm} (30)

This expression can be used to obtain different realizations and thus estimate the pdf of $z$, by Monte Carlo sampling of the random vector $\xi$. Each realization of $z$ represents the mean and covariance of the uncertainty distribution of $x$ corresponding to a particular realization of the uncertain model parameters. As noted earlier, for a given realization of the model parameters, the distribution of $x$ is Gaussian due to the additive white Gaussian forcing term. The complete distribution of the original states $x_i$ at any time step $k$, can thus be estimated from its realizations obtained by independent Monte Carlo sampling of the various Gaussian distributions resulting from the various samples of $\xi$.

In this approach, the pdf $\pi$ of the uncertain states $x_i(k, \xi, \omega)$ is obtained from the following:

$$
\pi(x_i(k, \xi, \omega)) = \int_{\Omega} \pi(x_i(k, \xi, \omega)|p(\xi)) \pi(p(\xi)) dp(\xi) \\
= \int_{\Omega} \pi(x_i(k, \xi, \omega)|\xi) g(\xi) d\xi
$$  \hspace{1cm} (31)

The mean of the solution $x_i(k, \xi, \omega)$ can be evaluated analytically using the equation:

$$
E[x_i(k, \xi, \omega)] = \mu(k) = \int_{\Omega} E[x_i(k, \xi, \omega)|\xi] g(\xi) d\xi \\
= \int_{\Omega} \left( \int_{-\infty}^{\infty} x_i(k, \xi, \omega) \pi(x_i|\xi) dx_i \right) g(\xi) d\xi
$$  \hspace{1cm} (32)

In this approach, for each realization of the model parameters $\xi$, the system states are Gaussian with the mean and covariance given by $z$. The first $n$ elements of $z(k, \xi)$ represent the mean, while the remaining $n(n+1)/2$ elements of $z(k, \xi)$ represent the covariance matrix of $x(k, \xi, \omega)$ for each realization of $\xi$. Letting,

$$x(k, \xi, \omega) = h(k, z(\xi), \omega), \text{ where } \omega \text{ is a Gaussian random vector } \in N(0, 1)$$

$$\mu(k) = \int_{\Omega} \left( \int_{-\infty}^{\infty} h(k, z(\xi), \omega) g(\omega) d\omega \right) g(\xi) d\xi
$$  \hspace{1cm} (33)

3.3 Equivalence and Convergence

The uncertainties in the stochastic dynamic model described by Eq. (17) are assumed to be independent. The solution $\pi(x_i(k, \xi, \omega))$ is given by Eq. (24) for the first approach and by Eq. (31) for the second.

(Doubts here!)
4 Results and Discussion

Some numerical examples are provided in this section to illustrate the two approaches for propagating uncertainty due to stochastic forcing on a model with uncertain parameters. The first example is a simple linear dynamic model with a Gaussian uncertain parameter and stochastic forcing.

4.1 Spring-Mass system

The example of a simple mass-spring-damper system with an uncertain spring stiffness coefficient $k$, driven by a zero mean Gaussian stochastic forcing $u$ is considered. The system is described by Eq. (34) and the evolution of the states of the system is shown in Fig. 2.

$$m \ddot{x} + c \dot{x} + k x = u$$  \hspace{1cm} (34)

![Fig. 2 Mass-spring-damper system](image)

The mass, $m = 1$, is released at $x_0 = 5$, with velocity $\dot{x}_0 = 0$. The system has a known damping constant $c = 0.1$. The spring stiffness $k$ is assumed to be Gaussian with mean 2 and standard deviation 0.5, while the zero mean Gaussian stochastic forcing has a standard deviation of 0.2. The true evolution of the uncertain states is obtained by Monte Carlo solution of the uncertain system Eq. (34), with 10000 sample runs of the model, independently sampling the uncertain parameter $k$ and the stochastic forcing. The evolution of these uncertainties in this linear dynamic model is considered using the two polynomial chaos (PC) approaches and compared with the Monte Carlo solution.

4.1.1 Propagation of the first two moments of the PC coefficients of the states

In this approach, the states of the given model are first approximated by a PC series expansion of order 7, representing the uncertainty in the model parameter $k$. Since $k$ is
Gaussian, Hermite polynomials are used as the basis functions in the PC approximation. The initial system is transformed into an expanded system with the PC coefficients as the new states. The uncertainty of the PC coefficients, due to the stochastic forcing, is propagated using the mean and covariance propagation equations of the linear
Kalman filter. In effect, the output obtained represents the uncertainty of the states characterizing the uncertainty in model parameters.

Fig. 3 shows the evolution of the means of the states. As noted earlier, the first term of the PC expansion of an uncertain variable represents its mean. In this example, the first state of the expanded system represents the mean position due to model parameter uncertainty. The uncertainty in this mean position, due to the stochastic forcing, is characterized by the mean and covariance propagation equations of the Kalman filter. This propagated mean position represents the true mean of the uncertain position, in the case of a linear model driven by Gaussian noise. It can be seen that the mean of the PC solution is consistent with that of the Monte Carlo solution, noted as truth in the figure. Fig. 4 shows the evolution of the entire probability distribution of the uncertain states, while Fig. 5 shows the probability distribution of the uncertain states at a particular time $t = 10$.

4.1.2 Propagation of the PC coefficients of the first two moments of the states

In this approach, the given model is replaced by an augmented model whose states represent the uncertainty of the original states due to stochastic forcing. The mean and covariance propagation equations represent the dynamics of this new model, accounting for the first two moments of the distribution of the actual states. The states of this augmented model are then approximated by a PC series expansion of order 7, representing the uncertainty in the model parameter $k$. Since $k$ is Gaussian, Hermite polynomials are used as the basis functions in the PC approximation. The system is then transformed into a deterministic system of equations with the PC coefficients of the augmented states, as the new states. In effect, the solution of this system represents the uncertainty of the states characterizing the uncertainty due to stochastic forcing.
Fig. 7 Evolution of pdf of uncertain states

Fig. 8 pdf of uncertain states at time t = 10

Fig. 6 shows the evolution of the means of the states. In this example, the first state of the augmented system represents the mean position due to stochastic forcing. This mean position represents the true mean of the uncertain position, in the case of a linear model driven by Gaussian noise. The uncertainty in this mean position, due to the uncertainty in the model parameter, is characterized by the PC series expansion using Hermite polynomials as basis functions. As noted before, the first term of the PC expansion of a state represents its mean. It can be seen that the mean of the PC solution is consistent with that of the Monte Carlo solution, noted as truth in the figure. Fig. 7 shows the evolution of the entire probability distribution of the uncertain states, while Fig. 8 shows the probability distribution of the uncertain states at a particular time t = 10.

5 Conclusion

Two approaches are proposed in this work for the accurate description of uncertainty evolution in linear dynamic models with parametric and initial condition uncertainties, and driven by AWGN. The output in the first approach is the uncertainty of states characterizing the uncertainty in model parameters, whereas the output in the second approach is the uncertainty of states characterizing the uncertainty in the forcing term. In both the methods, the uncertainty due to the AWGN stochastic forcing is propagated using the mean and covariance propagation equations and that due to uncertain model parameters using the polynomial chaos. While the mean and covariance propagation equations are exact only for white Gaussian stochastic forcing in linear dynamic models,
the polynomial chaos approach can be used for any probability distribution of model parameters. This approach can be directly extended to nonlinear dynamic models, where the uncertainty due to white noise can be approximately propagated using model linearization or unscented propagation approaches.

The polynomial chaos approach involves fewer computations than the standard Monte Carlo solution approach which requires solving the dynamical model many times for many realizations of the uncertain parameters. When measurements are available, this information about the distribution of the solution can be used to make accurate predictions using filtering techniques. This suggests an extension of this approach to robust filtering problems with dynamical models having known parametric uncertainty distributions, a task currently under investigation.

References


