Nonlinear Trajectory Generation Using Global Local Approximations

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Abstract—In this paper we propose a method that uses local approximations to construct a globally continuous function. Such a method increases the variability of the parameterized function space, which is useful in the context of numerically solving nonlinear trajectory generation problems.

I. INTRODUCTION
Trajectory generation has been an active area of research in very diverse fields. In robotics this is more popularly known as motion planning and several methods have been proposed [1]. In this paper we consider trajectory generation based on optimal control problem formulation. We are interested in determining trajectories in time that minimises a cost function and satisfies boundary constraints, path constraints and the system dynamics. Typically, the unknown trajectories are parameterized and the optimal control problem is solved as a nonlinear programming problem (NLP). Several methods [2] including direct collocation [3], [4], [5], pseudo-spectral methods [6], [7], [8] and spline approximations [9], [10], [11], [12] have been proposed. For computational efficiency it is desired that the feasible nonlinear trajectory is approximated by a few continuous basis functions with unknown parameters and the approximation has local support. A key question regarding the selection of these continuous basis functions is “How irregular is the feasible solution set?”. A globally valid set of basis functions should be sufficient if feasible solution set is well connected and all feasible solutions are globally smooth. The variability of a particular nonlinear system is especially non-uniform in space and time; some regions may be highly irregular and others may be smooth and linear. As a consequence, it is improbable that a globally nonlinear function can be guessed a priori that represents such phenomena accurately and efficiently. An alternative to global learning is local learning [13], [14] based upon divide and conquer strategy. Local learning lead naturally to localized adaptation of the approximation degree of freedom to represent the variations actually present. A potential downside of employing local approximation methods is that they may be computationally expensive as we need to solve the optimization problem in each local neighborhood.

In this paper we present a method to parameterise a globally smooth function based on local approximations. We first formulate the trajectory generation problem as an optimal control problem. Methods for constructing a globally smooth function from local approximations is presented next. This is followed by an outline of the process that transcribes optimal control problems to nonlinear programming problems. The paper concludes with results and summary.

II. PROBLEM FORMULATION & CURRENT SOLUTION APPROACHES
Consider the nonlinear system
\[ \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m. \]
It is desired to find a solution \( x(t) \) and the associated control \( u(t) \) that minimises the performance index
\[
J(x, u) = \Phi_i(x(t_0), u(t_0)) + \int_{t_0}^{t_f} L(x(t), u(t))dt + \Phi_f(x(t_f), u(t_f)),
\]
where \( L(x(t), u(t)) \) is a nonlinear function. The trajectories \( x(t) \) and \( u(t) \) are subject to initial, final and path constraints. This poses the trajectory generation problem as the optimal control problem:
\[
\min_{x, u} J(x, u)
\]
subject to dynamics
\[ \dot{x} = h(x, u) \]
and constraints
\[ l_i \leq \Psi_i(x(t_i), u(t_i)) \leq u_i \quad (\text{initial}) \]
\[ l_i \leq \Psi_i(x(t), u(t)) \leq u_i \quad (\text{path}) \]
\[ l_f \leq \Psi_f(x(t_f), u(t_f)) \leq u_f \quad (\text{final}). \]

Let us represent this optimal control problem as \( \mathcal{P} \). The posed problem \( \mathcal{P} \) is solved numerically by parameterising the unknown trajectories and transcribing it to a nonlinear programming problem of the form
\[
\min_{\rho} F(\rho)
\]
subject to
\[
L \leq \left\{ \begin{array}{c} \rho \\ C(\rho) \end{array} \right\} \leq U,
\]
where \( \rho \) represents the parameters used to describe the unknown trajectories. Many methods have been proposed to achieve this, including classical collocation methods, pseudo-spectral methods and spline approximations. Transcription based on direct collocation methods result in large problem size and is an inefficient representation of the optimal control problem. In pseudo-spectral methods, Legendre or Chebyshev polynomials are used as basis functions. The coefficients of these functions are the parameters of optimization. A deficiency in this approach is that the basis functions do not have local support and demonstrate the Gibbs phenomenon. In spline based methods, the trajectory is assumed to be a composite of polynomial pieces stitched together in a manner that ensures a certain degree of smoothness across the break points. Smoothness conditions can be enforced implicitly by using B-Splines as basis functions. The unknowns are the coefficients of the B-Splines. B-Splines offer local support and are approximating functions of choice when approximation is desired over a large interval. However, the approximation is done with polynomials of the same order and lacks variability.

In the proposed method, any function of choice can be used to locally approximate the trajectory. Smoothness of the local approximations across boundaries is ensured by means of a weighting function that satisfies the partition of unity condition. As an example, a trajectory could be assumed to be a composite of polynomials, exponentials and trigonometric functions. The resultant curve is a weighted sum of the pieces and the smoothness across boundary is ensured by designing the weighting function appropriately. Thus the freedom to incorporate any local approximations, to create a global trajectory, is the strength of the proposed approach and offers a more generalised framework for trajectory generation and allows greater variability.

III. GLOBAL-LOCAL APPROXIMATION

In this section, we present the algorithm that blends locally approximate functions to a piecewise globally continuous function. Let us assume,

1) \( \mathcal{T} = \{t_1, \cdots, t_n\} \) is a uniform grid with spacing \( h \) and \( -\infty < t_i < t_{i+1} < \infty \). Uniform grid is assumed for simplicity. It is not a limitation of the proposed approach.

2) \( \mathcal{F} = \{f_1(t), \cdots, f_n(t)\} \) is a set of continuous functions that approximates the global function \( f(t) \) at points \( t_i \in \mathcal{T} \).

3) \( \mathcal{W} \) is a set of continuous functions in the interval \([-1, 1] \).

We define a non-dimensional local coordinate \( \tau_i \in [-1, 1] \) as \( \tau_i = (t - t_i)/h \), centered on the \( i^{th} \) vertex \( t = t_i \). Given \( \mathcal{F} \) and weighting function \( w(\tau_i) \in \mathcal{W} \), the weighted average approximation is defined as:
\[
\tilde{f}_i(t) = w(\tau_i)f_i(t) + w(\tau_{i+1})f_{i+1}(t),
\]
for \( 0 \leq \tau_i, \tau_{i+1} < 1 \) and \( t \in [t_i, t_{i+1}] \).

The weighting function \( w(\tau_i) \) is used to blend or average the two adjacent preliminary local approximations \( f_i(t) \) and \( f_{i+1}(t) \). The global function is given by the expression,
\[
f(t) = \sum_{i=1}^{n} w(\tau_i)f_i(t), \quad t \in (-\infty, \infty), \tau_i \in [-1, 1].
\]

The preliminary approximations \( f_i(t) \in \mathcal{F} \) are completely arbitrary, as long as they are smooth and represents the local behavior of \( f(t) \) well. This leads to the question:

Is there a choice of weighting function that will guarantee piecewise global continuity while leaving the freedom to fit local data by any desired local functions?

The answer to this question turns out to be yes! In refs. [15], [16], it is shown that if the weighting functions of eqn.(1) satisfies the following boundary value problem, then the weighted average approximation in eqn.(1) form an \( m^{th} \)-order continuous globally valid model with complete freedom in the choice of the local approximations in \( \mathcal{F} \). These conditions characterise the set \( \mathcal{W} \). That is,
\[
\mathcal{W} = \{ w(\tau) : w^{(k)}(0) = 0, w^{(k)}(1) = 0, k = 0, \cdots, m \}
\]
\[
w^{(k)}(\tau) + w^{(k)}(\tau - 1) = 1, \quad \forall \tau, -1 \leq \tau \leq 1
\]
where \( w^{(k)} = \frac{d^k w}{d\tau^k} \). The conditions can be summarised as follows.
1) The first derivative of the weighting function must have an $m^{th}$-order osculation with $w(t) = 1$ at the centroid of its respective local approximation.

2) The weighting function must have an $(m + 1)^{th}$-order zero at the centroid of its neighboring local approximation.

3) The sum of two neighboring weighting functions must be unity over the entire closed interval between their corresponding adjacent local functional approximations.

If the weighting function is assumed to be a polynomial in the independent variable $\tau$, then adopting the procedure listed in ref. [17], [16], the lowest order weight function (for $m = 1$) can be shown to be

$$w(\tau) = \begin{cases} 
1 - \frac{\tau^2}{2}(3 + 2\tau), & -1 < \tau < 0 \\
1 - \frac{\tau^2}{2}(3 - 2\tau), & 0 \leq \tau \leq 1 
\end{cases}$$

and are shown in figure 1(a).

The weighting functions obtained by solving the boundary value problem are listed in Table I and have the following properties.

1) The domain of weight function $w(\tau)$ is a compact space,

2) $w(\tau) > 0$, $\forall \tau \in (-1, 1)$,

3) $w(\tau) = 0$, $|\tau| \geq 1$,

4) $w(\tau)$ is a monotonically decreasing function of $|\tau|$, $\forall \tau \in (-1, 1)$.

Observe that by choosing the weighting functions given by eqn.(2), we are guaranteed global piecewise continuity for all possible continuous local approximations in $F$. One retains the freedom to vary the degree of the local approximations as needed, to fit the local behavior of $f(t)$, and rely upon $w(\tau)$ to enforce continuity across knot points.

If the global function is desired to be identical to the local approximation over the interval $[t_i, t_{i+1}]$, then specifying the same local approximation at the extreme points of the interval guarantees it. This can be seen from eqn.(1).

$$f_i(t) = f_i(t_i) + w_1(t) f_{i+1}(t)$$

where $f_{i0}, f_{i+1}, \alpha, \beta, \gamma, \omega$ and $\phi$ are constants. With reference to fig.2(a), note that $f(t)$ is locally approximated by an exponential function at $A(t = 0)$, a linear function at $B(t = \frac{1}{3})$ and by a trigonometric function at $C(t = \frac{2}{3})$.

**Example 1** Let $f_1(t), f_2(t)$ and $f_3(t)$ be three functions that locally approximates $f(t)$, $\forall t \in [0, 1]$. Let the functions be:

$$f_1(t) = f_{10}e^{-\alpha t}$$

$$f_2(t) = f_{20} + \beta t$$

$$f_3(t) = f_{30} + \gamma \cos(\omega t - \phi)$$

where $f_{10}, f_{20}, f_{30}, \alpha, \beta, \gamma, \omega$ and $\phi$ are constants. With reference to fig.2(a), note that $f(t)$ is locally approximated by an exponential function at $A(t = 0)$, a linear function at $B(t = \frac{1}{3})$ and by a trigonometric function at $C(t = \frac{2}{3})$.

Let us assume that second order continuity is desired. Then the weighting functions $w(\tau_i)$ are as shown in fig.1(b). Figure 2(a) shows the local approximations $f_1(t), f_2(t)$ and $f_3(t)$. Observe that the approximated function osculates in value and derivative with the local approximations $f_1(t), f_2(t)$ and $f_3(t)$, at points $A, B$ and $C$ respectively. Also, note that the global approximation is identical to the local approximation in the interval $[\frac{2}{3}, 1]$, this is achieved by specifying the same local approximations at $t = \frac{2}{3}$ and $t = 1$.

This technique of blending local approximations to a piecewise globally continuous function can be extended to higher dimensions as well. In Ref. [16], this method is developed systematically and extended rigorously to approximation with arbitrary order continuity in an $n$ dimensional space.
TABLE I
WEIGHT FUNCTIONS FOR HIGHER ORDER CONTINUITY

| Order of Piecewise Continuity | Weight Function: $w(\tau), \forall \tau \in \{-1 \leq \tau \leq 1\}, \eta \Delta |\tau|$ |
|-------------------------------|------------------------------------------------|
| 0                             | $w(\tau) = 1 - \eta$ |
| 1                             | $w(\tau) = 1 - \eta^2(3 - 2\eta)$ |
| 2                             | $w(\tau) = 1 - \eta^4(10 - 15\eta + 6\eta^2)$ |
| 3                             | $w(\tau) = 1 - \eta^6(35 - 84\eta + 70\eta^2 - 20\eta^3)$ |
| ...                           | ... |
| $m$                           | $w(\tau) = 1 - \eta^{m+1} \left\{ \frac{(2m+1)!!(1-\eta)^m}{(m!)^2} \sum_{k=0}^{m} \frac{(-1)^k}{2m-k+1} \binom{m}{k} \eta^{m-k} \right\}$ |

IV. TRANSCRIPTION TO NLP USING GLOBAL-LOCAL APPROXIMATION

In the transcription process described here, it is assumed that the unknown trajectories in the optimization problem $P$ are $x(t)$ and $u(t)$. In some cases, the optimization problem can be reduced to a problem in a lower dimensional space, using techniques based on differential flatness [18], inverse dynamics optimization[19], etc. The transcription process presented here can be easily adapted to such problems.

Let us assume trajectories $x(t)$ and $u(t)$ are approximated by the following parameterized functions:

$$x(t) = \sum_{i=1}^{n} w_x(\tau_i) f_i(t, \rho_x) \tag{3}$$

$$u(t) = \sum_{i=1}^{m} w_u(\tau_i) g_i(t, \rho_u), \tag{4}$$

where $w_x(\tau_i)$ and $w_u(\tau_i)$ are the weighting functions and $f_i(t, \rho_x)$ and $g_i(t, \rho_u)$ are the parameterized local approximations of $x(t)$ and $u(t)$ respectively, where $\rho_x \in \mathbb{R}^{n_{\rho_x}}$ and $\rho_u \in \mathbb{R}^{n_{\rho_u}}$ with $n_{\rho_x}, n_{\rho_u} \in \mathbb{Z}^+$, the set of positive integers. Let

$$\rho = \left\{ \rho_x, \rho_u \right\} \in \mathbb{R}^{(n_{\rho_x} + n_{\rho_u})},$$

which represents the parameters of optimization.

The time derivative of $x(t)$ is

$$\dot{x}(t) = \sum_{i=1}^{n} \dot{w}_x(\tau_i) f_i(t, \rho_x) + w_x(\tau_i) \dot{f}_i(t, \rho_x),$$

which involves $\dot{w}_x(\tau_i)$ and $\dot{f}_i(t, \rho_x)$. These are easily computable.

Next we define collocation points. These are points where the cost function is evaluated and constraints are satisfied. It is not necessary to have the same set of collocation points for both cost and constraints, but we
will assume it to be so for simplicity. Let $\mathcal{T}_c = \{\xi_i\}, i = 0, \cdots, N$, be the set of collocation points where $\xi_0$ corresponds to the initial time $t_0$ and $\xi_N$ corresponds to the final time $t_f$. For approximating functions that are polynomials, the optimal distribution of $\xi_i$ can be determined using the appropriate quadrature formula.

Cost and constraint functions evaluated at the collocation points become functions of $\rho$. The dynamics of the system is an equality path constraint that needs to be satisfied at every collocation point, which becomes an equality constraint on $\rho$. The resulting nonlinear programming problem in $\rho$ can be solved using any commercially available software such as SNOPT [20] or NPSOL [21].

V. RESULTS

In this section we apply the proposed method to an example problem based on the Vanderpol oscillator. The optimization problem is defined as [12]:

$$\min_{x(t), u(t)} \int_0^5 \frac{1}{2} (x_1^2 + x_2^2 + u^2) dt$$

subject to system dynamics

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + u$$

and terminal constraints

$$x_1(0) = 1$$
$$x_2(0) = 0$$
$$x_2(5) - x_1(5) = 1$$

The problem can be reduced to a problem with one unknown, $z(t) \equiv x_1(t)$, and the optimization problem in terms of $z(t)$ is

$$\min_{z(t)} \int_0^5 (z^2 + \dot{z}^2 + [\ddot{z} + z - (1 - z^2)\dot{z}]^2) dt$$

subject to terminal constraints

$$z(0) = 1$$
$$\dot{z}(0) = 0$$
$$\ddot{z}(5) - z(5) = 1.$$ 

Let us assume, for academic reasons, that $z(t)$ is constraint to be approximated by $z_1(t) = a_0 + a_1 \cos(a_2 t)$ in the neighborhood of $t = 0$ and by $z_2(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4$ in the neighborhood of $t = 5$. The NLP problem in this case has total eight unknowns, namely $a_0, a_1, a_2, b_0, b_1, b_2, b_3, b_4$. The resulting NLP problem is solved by using the SNOPT optimization package.

Figure 3(a) shows the suboptimal trajectory $z^*(t)$. Notice that $z^*(t)$ oscillates $z_1(t)$ and $z_2(t)$ at the appropriate points in time. We chose $m = 3$ for the weighting functions. Figures 3(b) and 3(c) shows the first two time derivatives of the suboptimal trajectories i.e. $\dot{z}^*(t)$ and $\ddot{z}^*(t)$.

The suboptimal cost with these approximations is 1.7077 with 8 parameters. Using B-splines the suboptimal cost is 1.7022 with 7 parameters (2 intervals, $5^{th}$ order polynomials and $3^{rd}$ order smoothness). Increasing the number of intervals to 30 achieves the suboptimal cost 1.6859, with 63 parameters (30 intervals, $5^{th}$ order polynomials and $3^{rd}$ order smoothness) [12].

To do a detailed study and comparison with B-spline approximation we approximate trajectories in each local-interval with $3^{rd}$ order polynomial basis functions. We choose $m = 2$ for the blending weight functions to enforce the $3^{rd}$ order smoothness condition. Table II shows that the suboptimal cost decreases as the number of local approximations are increased. Also, it is clear from Table II numbers that the proposed method results in 25% decrease in cost with fewer parameters, as compared to the results presented in Ref. [12].

Finally, we fully appreciate the fact that the results from any test case are difficult to extrapolate and a detailed study is required to comment on the numerical efficiency and accuracy of the proposed algorithm. However, the results presented in this section do provide the compelling evidences and basis for optimism.

VI. SUMMARY

In this paper we presented a method for using local approximations to construct a globally continuous function. The resulting approximation also has local support. Thus it is amenable for parameterizing trajectory space for numerical solution of optimal control problems with high variability. We demonstrated the method on an academic problem based on the Vanderpol oscillator. Although the parameterized trajectories guarantee oscillation in value and derivatives with the local approximations, it may have large derivatives elsewhere. This can potentially lower bound the suboptimal solution. Trajectory generation using global-local approximation is an ongoing
research effort in our group and we hope to address this issue in future publications.

REFERENCES