A Quasi-Gaussian Kalman Filter

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Abstract—In this paper, we present a Gaussian approximation to the nonlinear filtering problem, namely the quasi-Gaussian Kalman filter. Starting with the recursive Bayes filter, we invoke the Gaussian approximation to reduce the filtering problem into an optimal Kalman recursion. We use the Fokker-Planck/ moment evolution equations for stochastic dynamic equations to evaluate the prediction terms in the Kalman recursions. We propose two methods, one based on stochastic linearization and the other based on a direct evaluation of the innovations terms, to perform the measurement update in the Kalman recursion. We test our filter on a simple two dimensional example, where the nonlinearity of the system dynamics and the measurement equations can be varied, and compare its performance to that of an extended Kalman filter.

I. INTRODUCTION

In this paper, we present a Gaussian approximation to the exact nonlinear filtering problem, namely the quasi-Gaussian Kalman filter. We consider a problem with continuous time state dynamics and discrete measurement updates. We use a Gaussian approximation to the nonlinear filtering problem in order to reduce the problem of nonlinear filtering to one of maintaining the estimates of the first two moments of the state variable based on the observations, i.e., an optimal Kalman recursion. We use the Fokker Planck equations/ moment evolution equations for stochastic dynamical systems in order to evaluate the prediction terms in the nonlinear filtering recursions. Then, we propose two methods, one based on stochastic linearization and another based on direct evaluation, to evaluate the innovations terms for forming the updates of the mean and covariance of the state variable. We test the proposed filter by testing it on a simple one dimensional example, where the nonlinearity of the system dynamics and the measurement equations can both be varied, and compare its performance to that of an extended Kalman filter (EKF).

Nonlinear filtering has been the subject of intensive research ever since the publication of Kalman’s celebrated work in 1960 [1]. It is known that the solution to the general nonlinear filtering problem results in an infinite dimensional filter in the sense that an infinite system of differential equations need to be solved in real time in order to solve the nonlinear filtering problem exactly [2]–[5]. Hence, the nonlinear filtering problem is computationally tractable only if we consider finite dimensional approximations of the true infinite dimensional filter [6]–[10]. However, even these methods suffer from the fact that the computations involved are so involved that their implementation in real time is infeasible.

The Bayes filter [4] offers the optimal recursive solution to the nonlinear filtering problem. However, the implementation of the Bayes filter in real time is computationally intractable because of the multi-dimensional integrals involved in the recursive equations. We note however that researchers have reported attempts at tackling this problem for low-dimensional systems [11], [12]. It is known that the implementation of the Bayes filter is rendered tractable in the case when all the involved random variables are assumed to remain Gaussian [13], [14], the obtained filter reducing to the Kalman filter in the case of linear dynamics and measurements. The primary concern of this class of methods is to predict the mean and covariance of the state variable between measurement updates, while evaluating the covariance of the measurement and the cross-covariance between the state and the measurement after an update, in order to evaluate the prediction and measurement update terms in a Kalman-type recursion. The extended Kalman Filter (EKF) [13], [14] linearizes the system dynamics about the current estimate of the state and the measurement equations about the current predicted estimate of the state variable, in order to evaluate the prediction and measurement update terms in the optimal Kalman recursive equations. The EKF by far has been the most popular nonlinear filter in practical applications in past several decades. In recent years, there has been renewed interest in the topic of nonlinear filtering with the discovery of the “sigma-point” Kalman filters [15]–[18], [20]. These nonlinear filters consider the nonlinear filtering problem in a discrete time setting. The different terms of interest in the optimal Kalman recursions are obtained using a semi-global technique known as stochastic or statistical linearization [19]. The method essentially consists of discretizing the domain of the random variable into a set of weighted sigma points and transforming these through the nonlinear map in order to obtain the distribution characteristics of the transformed random variable. The various sigma point algorithms differ from each other in their choice of the sigma points, for instance the unscented Kalman filter (UKF) [15], the central difference Kalman filter (CDKF) [21], the square root form UKF and CDKF [20] and so on. In our approach to the nonlinear filtering problem, we use a Gaussian approximation on the random variables of interest. However, we use a continuous time model for the system dynamics while assuming that the
measurements are made in discrete time. This allows us to use the elegant formalism of the Fokker-Planck-Kolmogorov equations [22], [23] to evaluate the prediction terms in the nonlinear filtering recursions. We use two methods: one based on stochastic linearization and the other based on direct evaluation of the measurement update terms, in order to evaluate the innovations terms in the optimal recursions, i.e., to linearize the measurement equations. Our method does not require the choice of a set of sigma points and thus, is free from the design aspect of the sigma point Kalman filters. However, central to the implementation of our filter is the evaluation of integrals of the form of a product of a function and a Gaussian function. In the case of polynomial nonlinearities, these are the higher order moments of a Gaussian random variable. The rest of the paper is organized as follows.

In section 2, we introduce the Bayes filter. In section 3, we dwell on the topic of Gaussian approximations in nonlinear filtering problems and present the optimal Kalman recursions. In section 4, we present our quasi-Gaussian Kalman filter and the methods we use to evaluate the optimal terms in the Kalman recursions for the approximate nonlinear filtering problem. In section 5, we present a one dimensional example, with variable nonlinearity in both system dynamics and measurement, and compare the performance of our filter with that of an EKF. In section 6, we present our conclusions and directions for further research.

II. BAYES FILTER

In this section, we introduce the recursive Bayes filter. Consider a system whose dynamics are governed by the Stochastic Ito differential equation,

$$dx = f(x)dt + g(x)dW$$

(1), and \( p(y_k/x_k) \) represents the probabilistic observation model corresponding to the observation equation (2). The first equation above represents the prediction step of the filter and the second equation corresponds to the measurement update step of the filtering equations. The prediction step in the Bayes filter uses the Chapman-Kolmogorov equations [22] to propagate the prior probability density of the state in between the measurement updates while the measurement update step uses Bayes rule to update the predicted prior based on the observed measurement. Note that evaluating the multi-dimensional integral in the above equations is a formidable challenge in general. The predicted probability density function, \( p(x_k/F^{k-1}) \), can also be obtained by solving the Fokker-Planck-Kolmogorov equation [22] corresponding to the Ito differential equation (1) given by,

$$\frac{\partial p}{\partial t} = - \sum_i f_i \frac{\partial p}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 (gQg)_{ij}}{\partial x_i \partial x_j}$$

(5)

where \( Q \) represents the intensity of the Wiener process \( W \).

Let

$$L_{FP} \equiv - \sum_i f_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 (gQg)_{ij}}{\partial x_i \partial x_j}$$

(6)

where \( L_{FP} \) denotes the differential operator corresponding to the Fokker-Planck-Kolmogorov equation. Thus, the integral in the Bayes filter corresponding to the prediction equation (3) can be evaluated by solving the above partial differential equation. The prediction step in the Bayes filter can be solved by either evaluating the integral in the prediction equation through a Monte-Carlo type method (the subject matter of particle filters [24]) or by solving the above partial differential equation. Note that the Fokker Planck equation is a linear partial differential equation. However, analytical solutions for the Fokker Planck equations are very rare and numerical methods have to be used in order to solve them. Hence, implementation of the Bayes filter in real time is computationally intractable except in the linear-Gaussian case, which leads to the celebrated Kalman Filter [1]. In this case, all the corresponding random variables are also Gaussian and thus, only estimates of the mean and covariance of the state need to be maintained in order to obtain the optimal state estimates. However, the same is not true for the nonlinear case and in general, an infinite dimensional filter is obtained in the sense that estimates of all the higher order moments of the state variable, or equivalently the probability density of the state variable, need to be maintained in order to obtain the optimal estimates of the system state. In such cases, a finite dimensional approximation is usually made to the infinite dimensional filter. The most widely used such finite dimensional filter is the extended Kalman Filter (EKF). Before we get to the details of the Kalman Filter, we shall first investigate the topic of Gaussian approximations in filtering problems.
One method for obtaining a finite dimensional approximation to the infinite dimensional filter corresponding to a general nonlinear filtering problem is to make the approximation that the state, noise and the observation terms in the general nonlinear filtering problem can all be approximated by Gaussian random variables. Then, the Bayesian recursion introduced in the previous section can be greatly simplified since it is known that the posterior distribution of the state variable remains Gaussian under these approximations. Thus, estimates of only the conditional mean, \( E(x_t | \mathcal{F}_t) \), and the covariance, \( P_x \), need to be maintained in order to obtain the optimal estimates of the system state. It can be shown that under the Gaussian approximation, we obtain following recursive filter [4], [13]:

\[
\bar{x}_k = \bar{x}_k^- + L_k(y_k - y^*_k), \quad (7)
\]

\[
P_{x_k} = P_{x_k}^- - L_k P_y L_k^t, \quad (8)
\]

\[
L_k = P_{x_k y_k} P_{y_k}^{-1}, \quad (9)
\]

where \( \bar{x}_k \) and \( \bar{x}_k^- \) represent the predicted and updated values of the mean of the random vector \( x_k \), \( P^-_x \), and \( P_{x_k} \) represent the predicted and updated values of the covariance matrix of \( x_k \), \( P_{y_k} \) is the covariance matrix of the observation \( y_k \) and \( P_{x_k y_k} \) is the cross-covariance matrix between the state \( x_k \) and observation \( y_k \). The key to implementing the above recursion is to evaluate the quantities \( \bar{x}_k^- \), \( P^-_x \), \( P_{y_k} \) and \( P_{x_k y_k} \), i.e., the prediction and measurement update steps of the nonlinear filtering equations. Different filters use various different ways to evaluate the above mentioned quantities.

In the case of the EKF, the predicted mean and covariance of the variable \( x_k \) are evaluated by linearizing the system dynamics about the previous estimate \( x_{k-1} \) and using linear Gaussian analysis to propagate the mean and the covariance, while the quantities \( P_{y_k} \) and \( P_{x_k y_k} \) are evaluated by linearizing the measurement equation about the current predicted value of \( x_k \) and using linear Gaussian analysis.

In recent years there has been a lot of interest in sigma point kalman filters which use statistical linearization methods to find out the quantities of interest in the nonlinear filtering recursive formulas [17], [20]. The sigma point filters perform the linearization of the system dynamics and the measurement equations in a statistical sense as opposed to the Jacobean linearization which is performed in the case of the EKF.

In our approach, we adopt a different approach to evaluate the prediction and innovation terms in the filtering recursions. We use the Fokker Planck equations/ moment evolution equations for a stochastic dynamical system, along with the Gaussian approximation, to evaluate the prediction terms and then perform a stochastic linearization of the measurement equation in order to evaluate the innovation terms. The details of the method are presented in the next section.

### IV. A Quasi-Gaussian Kalman Filter

In this section, we present our approach to nonlinear filtering based on Gaussian approximations on the prediction and innovation terms in the filtering equations. As mentioned previously, we use the Fokker Planck equations/ Moment evolution equations for stochastic dynamical systems, along with the Gaussian approximation on all the random variables involved in the filtering problem, to evaluate the prediction terms in the filtering recursions. Then we propose two methods to evaluate the innovation terms: (a) stochastic linearization to linearize the measurement equations and (b) a direct method to evaluate the innovations terms.

In the following subsections, we shall present the methods for evaluating the prediction and innovation terms in the filtering equations.

#### A. Prediction

In this section we shall present the method for evaluating the prediction terms in the nonlinear filtering equations. Recall the nonlinear filtering equations (7)-(9). We shall now present our method for evaluating the prediction terms involved in the same. Note that,

\[
\frac{\partial p}{\partial t} = L_{FP}(p), \quad (10)
\]

where \( L_{FP} \) is the differential operator corresponding to the Fokker Planck Kolmogorov equation or the Fokker Planck (FP) operator, and \( p(x) \) represents the probability density of the state variable. We shall find differential equations corresponding to the time evolution of the first and second moments of the state variable given the prior probability distribution of the state. Let the prior probability distribution at the \( k^{th} \) instant be given by \( N(\bar{x}_k, P_{x_k}) \). Note that we need to maintain only the first and second moments of the state variable because of the Gaussian approximation on the system. Let the mean and the correlation of the state vector between the time instants be denoted by \( \bar{x} \) and \( R_x \) respectively. By definition, we have the following:

\[
\bar{x} = \int x p(x) dx, \quad (11)
\]

\[
R_x = \int xx' p(x) dx, \quad (12)
\]

where \( p(x) \) is the probability distribution of the state variable \( x \). Then, it follows that the first and second moments of the state \( x \) evolve according to the following differential equations:

\[
\dot{\bar{x}} = \frac{d}{dt} \int x p(x) dx = \int x \frac{\partial p}{\partial t} dx = \int x L_{FP}(p(x)) dx, \quad (13)
\]

\[
\dot{R}_x = \frac{d}{dt} \int xx' p(x) dx = \int xx' L_{FP}(p(x)) dx, \quad (14)
\]

with the initial conditions \( \bar{x}_k \) and \( R_{x_k} \) respectively. In order to evaluate the predicted mean and covariance, the above differential equations need to be integrated between the \( k^{th} \) and \( (k+1)^{th} \) time instants. Also, due to the Gaussian approximation on the state variable, we approximate the probability
density function \( p(x) \) by \( N(\bar{x}, R_x) \). In general, evaluation of the right hand side in the above equation involves higher moments of the random variable \( x \) because of the Fokker Planck operator. The above equations are quite complicated to use in practice because of the messy nature of the Fokker Planck operator and thus, we present below an alternate method for deriving the moment evolution equations. We derive the differential equation governing the evolution of the correlation of the state \( x \), the equation governing the evolution of the mean can be derived in a similar fashion. Consider the stochastic Ito differential equation governing the dynamics of the system,

\[
dx = f(x)dt + g(x)dW. \tag{15}\]

We shall derive the differential equation governing the evolution of the correlation matrix in the sequel, the differential equation governing the evolution of the mean can be derived in a similar fashion. Then consider an infinitesimal change in the correlation of the state \( x \), \( dE( xx^t ) \). We have

\[
dE( xx^t ) = E[(x + dx)(x + dx)^t] - E[ xx^t ]. \tag{16}\]

Let \( dx = y \). Then, by definition we have the following:

\[
E[(x + y)(x + y)^t] = \int (x + y)(x + y)^t p(x, y) dx dy, \tag{17}\]

\[
E[ xx^t ] = \int xx^t p(x) dx, \tag{18}\]

where \( p(x) \) represents the distribution function of \( x \) and \( p(x, y) \) represents the joint distribution function of the random variables \( x \) and \( y \). We can write the following:

\[
E[ xx^t ] = \int xx^t p(x) dx = \int xx^t (\int p(y/x) dy) p(x) dx = \int xx^t p(x, y) dx dy, \tag{19}\]

where the last equality follows from the definition of conditional probabilities. Hence, it follows that

\[
dE[ xx^t ] = \int (xy^t + yx^t + yy^t) p(x, y) dx dy. \tag{20}\]

It can be shown after some work that the following hold:

\[
\int xy^t p(x, y) dx dy = dt \int xf^t(x)p(x) dx = E[xf^t(x)] dt, \tag{21}\]

\[
\int yx^t p(x, y) dx dy = E[f(x)x^t] dt. \tag{22}\]

Hence, we have the following:

\[
dE[ xx^t ] = (E[xf^t(x)] + E[f(x)x^t] + E[g(x)Qg^t(x)]) dt + (E[xf^t(x)]) dt^2. \tag{24}\]

Dividing by \( dt \) and neglecting the \( O(dt) \) term leads us to the following differential equation governing the evolution of the correlation of the random variable \( x \):

\[
\dot{E}( xx^t ) = E[xf^t(x)] + E[f(x)x^t] + E[g(x)Qg^t(x)]. \tag{25}\]

Similarly we can derive the evolution equation of the mean of the random variable \( x \):

\[
\dot{E}( x ) = E[f(x)]. \tag{26}\]

Hence, the predicted mean and covariance can be obtained by integrating the following differential equations between the measurement updates

\[
\dot{x} = E[f(x)], \tag{27}\]

\[
\dot{R}_x = E[f(x)x^t] + E[xf^t(x)] + E[g(x)Qg^t(x)], \tag{28}\]

where the initial conditions for the above differential equations is given by \((x_k, R_{xk})\). Due to the Gaussian approximation on the system the expectations in the above equations are evaluated by substituting \( p(x) = N(\bar{x}, R_x) \). If the functions \( f \) and \( g \) are polynomials or can be approximated sufficiently well using polynomials, the right hand side in the above equations is of the form of a product of a polynomial and a Gaussian function, and thus correspond to higher order moments of the state vector \( x \).

B. Measurement Update

In this section, we shall present two methods for evaluating the innovations terms in the nonlinear filtering equations. We use stochastic or statistical linearization in order to linearize the measurement equation and evaluate the innovations terms using linear Gaussian methods or we use a direct method for evaluating the innovations terms. First we shall give a brief overview of stochastic linearization.

1) Stochastic Linearization: Consider a general nonlinear vector function, \( h(x) \), of a Gaussian random vector \( x \). Let \( h(x) \in \mathbb{R}^m \) and \( x \in \mathbb{R}^n \). The stochastically linearized gain for the function \( h \) is given by \( K_{eq} \) where

\[
K_{eq} = \arg \min_K E[||h(x) - Kx||^2]. \tag{29}\]

Please refer to [25] for literature on statistical linearization. It can be shown that the equivalent gain \( K_{eq} \) is given by solving the following system of linear equations

\[
\int x_i h_j(x) g(x) dx = \sum_{m=1}^{n} k_{im} \int x_j x_m g(x) dx, \forall i, j; \tag{30}\]

\[
i = 1,.., m \text{ and } j = 1,..n. \text{ Thus, in the above equation we have obtained an expression for the equivalent gain for any general nonlinear function of a random vector } x. \text{ In the following, we use this equivalent gain to form the innovations terms for the nonlinear filter.}

In order to evaluate the innovation terms in the nonlinear filtering equations, we need to evaluate the measurement covariance matrix, \( P_{xx} \), and the state-measurement cross-covariance matrix, \( P_{x_ky_k} \). Using the equivalent gain for the nonlinear measurement function \( h(x) \) according to the above expression leads us to the following approximate expressions...
Hence, we can write the nonlinear quasi-Gaussian filtering
the observation and the cross covariance between the state
tribution of the state at the
\(k\)
The above equations are evaluated using the predicted dis-
and
\(K_{eq}\) in general is time varying.

2) Direct Method: In this section, we present a direct
method for the evaluation of the covariance matrix of the
measurements and the cross-covariance between the state and
the observations. Recall that the measurement equations are
given by the following equation:
\[ y_k = h(x_k) + v_k, \]  
(33)
where the noise \(v_k\) is uncorrelated from the state \(x_k\). We need
to evaluate the covariance matrix of the observations \(P_{yk}\) and
the cross covariance matrix between the observations and the
state, \(P_{x_ky_k}\), in order to evaluate the innovations terms in the
filtering equation. First, we shall evaluate the covariance of
the observations, \(P_{yk}\). Note that
\[ \bar{y}_k = E[y_k] = E[h(x_k) + v_k] = E[h(x_k)], \]  
(34)
since \(v_k\) is zero-mean and independent of the state \(x_k\). Next,
it can be shown that
\[ P_{yk} = E[h(x_k)h^t(x_k)] - \{E[h(x_k)]\}^2 + R_v, \]  
(35)
and
\[ P_{x_ky_k} = E[x_kh^t(x_k)] - \bar{x}_kE^t[h(x_k)]. \]  
(36)
The above equations are evaluated using the predicted dis-
tribution of the state at the \(k^{th}\) instant, \(x_k\). Thus, we have
the following expressions for evaluating the covariance of the
observation and the cross-covariance between the state and
the observation:
\[ P_{yk} = E[h(x_k)h^t(x_k)] - \{E[h(x_k)]\}^2 + R_v, \]  
(37)
\[ P_{x_ky_k} = E[x_kh^t(x_k)] - \bar{x}_kE^t[h(x_k)]. \]  
(38)

Hence, we can write the nonlinear quasi-Gaussian filtering
recursions as follows:

- **Stochastic linearization based:**
\[ \dot{x} = E[f(x)], \]  
(39)
\[ \dot{R}_x = E[f(x)x^t] + E[f(x)Qg^t(x)], \]  
(40)
\[ P_{yk} = K_{eq}^tP_{x_k}^{-1}K_{eq} + R_v, \]  
(41)
\[ P_{x_ky_k} = P_{x_k}^{-1}K_{eq}^t, \]  
(42)
where \(K_{eq}\) is obtained by solving the system of equations
(30).

- **Direct evaluation based:**
\[ \dot{x} = E[f(x)], \]  
(43)
\[ \dot{R}_x = E[f(x)x^t] + E[f(x)x^t] + E[g(x)Qg^t(x)], \]  
(44)
\[ P_{yk} = E[h(x_k)h^t(x_k)] - \{E[h(x_k)]\}^2 + R_v, \]  
(45)
\[ P_{x_ky_k} = E[x_kh^t(x_k)] - \bar{x}_kE^t[h(x_k)]. \]  
(46)

The initial conditions for the differential equations in the
predictions equations in the above filtering equations are
given by the mean and correlation of the state \(x\) at the
\((k-1)^{th}\) time instant and the differential equations are
integrated between the \((k-1)^{th}\) and the \(k^{th}\) instants to obtain
the predicted distribution of \(x_k\), which is in turn used in the
innovations equations.

In the above filtering equations, note that evaluation of both
the prediction and measurement update terms involves the
evaluation of the moments of a Gaussian random vector,
in the case of polynomial nonlinearities. Thus, the real-
time implementation of these filters would be critically
dependent on the efficient computation of these moments.

In the next section, we present an illustrative one dimensional
example and compare the performance of the quasi-Gaussian
Kalman filter presented in this section to that of an EKF.

V. ILLUSTRATIVE EXAMPLE

In this section, we provide a comprehensive comparison of
Quasi Linear Gaussian Kalman Filter (QGKF) algorithms
with conventional Extended Kalman Filter (EKF) algorithm.
These results demonstrate that how the nonlinear propagation
of mean and covariance of the assumed Gaussian p.d.f
significantly enhances the performance of the EKF algorithm.
For testing purpose, we consider the following nonlinear duffing
oscillator:
\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -\alpha x_1 - \beta x_1^3 - \gamma x_2 + w \]  
(47)
where \(w\) represents the process noise vector with covariance
matrix \(Q\). The truth measurements are assumed to be total
energy of the system given by following equation:
\[ \tilde{y} = \alpha x_1^2 + \beta x_1^4 + \gamma x_2^2 + \nu \]  
(48)
Here, \(\nu\) represents the measurement noise vector with as-
associated covariance matrix, \(R\). Note that the factor \(\beta\) can
be varied to increase or decrease the nonlinearity of the
dynamics and the measurement equations. Also the factor
\(\gamma\) controls how fast the system achieve steady state. We
mention that both of these factors \((\beta,\gamma)\) play an important
role in the estimation of state vector \(x = \{x_1 \ x_2\}\) from
Eqs. (47) and (48). To evaluate the performance of the QGKF
filters over the conventional EKF algorithm, we consider two
test cases:

1) Low nonlinearity case with \(\beta = 0.25\)
2) High nonlinearity case with \(\beta = 5\).

The various other simulation parameters for both the test
cases are listed in Table I. All the simulations are performed
for 200 seconds by sampling the measurement data every 1
second.

In Fig. 1, we show the performance of the different filters
for low nonlinearity case i.e. Case 1 with \(\beta = 0.25\). Fig. 1(a)
shows the plot of true system trajectory and various estimated
trajectories. Further, Figs. 1(b), 1(c) and 1(d) show the plots
of state estimation error for EKF, QGKF with stochastic
linearization and QGKF with direct evaluation, respectively.
Fig. 1. Simulation Results for Case 1

(a) True and Estimated States  
(b) Estimated EKF States  
(c) Estimated QGKF (SL) States  
(d) Estimated QGKF (DE) States  
(e) Diagonal Components of State Estimation Error Covariance Matrix  
(f) Off-Diagonal Components of State Estimation Error Covariance Matrix

Fig. 2. Simulation Results for Case 2

(a) True and Estimated States  
(b) Estimated EKF States  
(c) Estimated QGKF (SL) States  
(d) Estimated QGKF (DE) States  
(e) Diagonal Components of State Estimation Error Covariance Matrix  
(f) Off-Diagonal Components of State Estimation Error Covariance Matrix
From these figures, it is clear that all 3 filters converges well and state estimation errors lie with in corresponding 3-σ outliers. Figs. 1(e) and 1(f) show the plot of various components of state estimation covariance matrix for all 3 filters. From this figure, it is clear that although all 3 filter converges well but state estimates confidence bound is higher for QGKF filters than EKF filter. Now, in Fig. 2, we show the performance of the different filters for high nonlinearity case i.e. Case 2 with $\beta = 5$. Fig. 2(a) shows the plot of true system trajectory and various estimated trajectories. and Figs. 2(b), 2(c) and 2(d) show the plots of state estimation error for EKF, QGKF with stochastic linearization and QGKF with direct evaluation, respectively. From these figures, it is clear that all 3 filters converges well and state estimation errors lie with in corresponding 3-σ outliers. However, it should be noticed that estimation error for EKF are a order of magnitude higher than in case of both the QGKF filters. Further, Figs. 2(e) and 2(f) show the plot of various components of state estimation covariance matrix for all 3 filters. From this figure, it is clear that state estimation error covariance for EKF start diverging after 120 second. The high estimation errors and divergence of state estimation error covariance in case of EKF can be attributed to the high nonlinearity and low damping coefficient $\gamma$. Also, from Figs. 2(c) and 2(d), it is clear that the transient response of QGKF filter based upon direct evaluation is marginally better than the transient response of stochastic linearization based QGKF filter.

In this section we have presented a two dimensional filtering example where the nonlinearity of the system dynamics and the measurement equations were both varied and the performance of the QGKF was compared to that of an EKF. It was seen that for the case of high and low nonlinearity in system dynamics and measurement equations, the QGKF always outperforms the EKF. Finally, we mention that although we have not proved the positive definiteness of updated state covariance matrix in case of QGKF but we never face the problem of semi-definiteness of covariance matrix.

VI. Conclusions

In this paper, we have presented a quasi-Gaussian Kalman filter for continuous dynamical systems. We have shown that if a Gaussian approximation is made on all the random variables involved in the filtering example, only estimates of the mean and covariance of the state need to be maintained in order to obtain optimal estimates of the system state. We have used the Fokker Planck equations/ moment evolution equations for a stochastic continuous dynamical system in order to evaluate the prediction terms involved in the nonlinear filtering recursions. In order to evaluate the innovations terms, we have proposed two different methods, one based on stochastic linearization and another based on direct evaluation of the measurement covariance and the state measurement cross covariance matrices. We have implemented the filters on a duffing oscillator example and compared the performance to that of an EKF.

It was noted that the complexity of the calculations involved in evaluating the prediction and innovations terms are of the same order as that of finding any general higher order moment of a Gaussian random vector when the nonlinearities are polynomial type. As such these can be efficiently calculated using Gauss-Hermite quadratures or the Wick formulas. Further, unlike in sigma point filters there is no design criterion for forming the set of sigma points to evaluate the prediction or the innovations terms. In general, higher order moments of the state variable are involved in evaluating the prediction and innovations terms which seems to lead to better performance of the filter. Also, given the dynamics and the measurement equations, the evaluation of these higher order moments can be “hardwired” in order to implement the filter in real time. However, any real time application of this filter would be dictated by the efficiency with which the moment calculations can be accomplished. Hence, there is need for further research into this aspect of the problem.

We also mention that proving the the positive definiteness of the updated covariance matrix for a nonlinear system (using any filtering algorithm for that matter) is an open problem in the field of filtering theory. This is a significant drawback with the proposed filter also as it can lead to oscillatory/ unstable behavior on part of the filter. However, to the best of our knowledge and experience, only measurement nonlinearity effects this aspect of the filter performance and the filter has no significant performance issues when the measurement equations are linear. In spite of this fact, it is seen from the example that the QGKF performs much better than the EKF even for highly nonlinear system and we never encounter the problem of semi-definiteness of QGKF. Thus, the stability properties of the proposed filter could be another direction of further research. Finally, we fully appreciate the truth that results from any test are difficult to extrapolate, however, testing the new filtering algorithm on a reasonable nonlinear system and providing comparisons to the most obvious competing algorithms does provide compelling evidence and a basis for optimism.

REFERENCES