Orthogonal Global/Local Approximation in N-dimensions: Applications to Input-Output Approximation

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Abstract

Several approximation ideas are presented. The main idea discussed is a weighting function technique that generates a global family of overlapping preliminary approximations whose centroids of validity lie on at the vertices of an N-dimensional grid, with vertices separated by a uniform step $h$. These preliminary approximations are constructed so they represent the behavior in hypercubes with a volume $(2h)^N$ centered on a typical vertex in the grid. These volumes, where the preliminary approximations are valid, overlap by 50% and are averaged in the overlapped $h^N$ volume hypercubes, interior to each contiguous set of $2^N$ vertices, to determine final approximations. We establish an averaging method that ensures these final approximations are globally piecewise continuous with adjacent approximations determined in an analogous averaging process, to some prescribed order of partial differentiation. The continuity conditions are enforced by using a unique set of weighting functions in the averaging process. The weight functions are designed to guarantee the global continuity conditions while retaining near complete freedom on the selection of the generating local approximations. However, if the preliminary local approximations are chosen as linear combinations of a set of basis functions constructed such that they are orthogonal with respect to the weight functions, then many advantages are realized, as demonstrated in the paper. Construction of the new set of orthogonal polynomials, and several properties of these functions are novel results presented in this paper. This paper enables a first: piecewise continuous least square approximation in N-dimensions, using orthogonal functions. Several applications are given which provide a basis for optimism on the practical value of the ideas presented.

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Introduction

To motivate this paper, consider Figure 1. Here we have 64,000 noisy measurements of an irregular function $F(x,y)$. These happen to be stereo ray intersection measurements from correlation of stereo images of topography near Ft. Sill, Oklahoma; however, they could be measurements of any complicated, irregular function for which a single global algebraic expression would likely be intractable. Suppose that it is desired to obtain a smooth, least square approximation of this function, perhaps with additional constraints imposed (e.g., in this case, the stereo correlation measurement process fails reliably over water, so the large spurious noise spikes over lakes Latonka and Elmer Thomas could be replaced by a constraint that the lake surface be at a known elevation). In lieu of a single global and necessarily complicated function, it is desired to represent the function using a family of simpler local approximations. Such local approximations would be much more attractive basis for local analysis. Alternatively, you may think of the local approximations as local Taylor series approximations (evaluated about local expansion points on a grid), or as any local approximations obtained from local measurements. However, if the local approximations are introduced without taking particular care, they will virtually certainly disagree in the value estimated for $F(x,y)$ and the derivatives thereof at any arbitrary point, although the discrepancies may be small. In other words, global continuity is not assured, unless we introduce methodology to guarantee the desired continuity properties. These challenges are compounded in higher dimensions, if usual approximation approaches are used. It is desired to determine a piecewise continuous global family of local least squares approximations, while having freedom to vary the nature (e.g., degrees of freedom) of the local approximations to reflect possibly large variations in the roughness of $F(x,y)$. While we are introducing the ideas in the setting of a data-fitting problem in a two dimensional space, the results are shown to be of much broader utility, and to generalize fully to approximation in an $N$ dimensional space.

From Reference [8], we summarize some features of the weighting function approach to approximation in two dimensions in Figure 2. From this figure, we introduce several qualitative observations: Notice the attractive properties of the weight functions: At any of the four vertices, we see the weight function (associated with the function whose centroid of validity is a given vertex) is unity, while the other three weight functions are zero at that vertex. Notice further that the weight functions have a qualitative bell shape, but fairing into a square base, the zero contour being the boundary opposite (e.g., 2-3-4) to the vertex (e.g., point 1) where the weight has a unit value. The four overlapping weight functions are a partition of unity, they add to unity everywhere in the overlapping unit region (which must be the case for an unbiased approximation). Furthermore, notice that along any boundary, only the two weight functions associated with the two approximations centered at the end points of that boundary are non-zero along that boundary, while the other two weight functions are zero (the partial derivatives of the other two weight functions are also along this boundaries). These continuity arguments on the averaged approximation of the func-
tion can be extended readily to corresponding properties on their partial derivatives: The averaged approximation osculate in value and partial derivatives with the four preliminary approximations at their corresponding vertices, and the function and both partial derivatives along any boundary are a weighted average of the corresponding two functions associated with the end point of that boundary and their partial derivatives are likewise an average of the partial derivatives of the functions at the end point of that boundary. Collectively, these observations lead to rigorous piecewise continuity of the averaged approximations, while leaving the user free to choose any preliminary local approximations desired or needed. These qualitative observations will be developed systematically in the subsequent sections and extended rigorously to approximation with arbitrary order continuity in an $N$ dimensional space.

**Approximation in 1-2- and $N$- Dimensions Using Weighting Functions**

The essential ideas can be introduced in a one-dimensional piecewise approximation problem. With reference to Figure 3, we discuss the one-dimensional problem. An arbitrary set of knots \{$^1X, ^2X, \ldots, ^kX, \ldots$\} (vertices) are introduced at a uniform distance $h$ apart; obviously, a non-dimensionalization of $x$ is introduced as a local coordinate $-1 \leq \frac{l}{h} \leq 1$; centered on the $l^{th}$ vertex $X = ^lX$. The

Figure 1: Approximation of Irregular Functions in Two Dimensions
Preliminary Approximations:

$F_{11}(x, y)$

$F_{01}(x, y)$

$F_{00}(x, y)$

$F_{10}(x, y)$

Weight functions:

$w_{11}(x, y) = x^2 (3 - 2x) y^2 (3 - 2y)$

$w_{01}(x, y) = w_{11}(1 - x, y)$

$w_{00}(x, y) = w_{11}(1 - x, 1 - y)$

$w_{10}(x, y) = w_{11}(x, 1 - y)$

Final Approximation:

$\tilde{F}(x, y) = \frac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{1} w_{ij}(x, y) F_{ij}(x, y)$

valid over $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$

Weight functions are a partition of unity:

$\sum_{i=0}^{1} \sum_{j=0}^{1} w_{ij}(x, y) = 1$

Figure 2: Qualitative Representation of the Averaging Process in Two Dimensions
weighted average approximation is introduced as

\[ F_I(X) = w(Ix)F_I(X) + w(Ix+1)F_{I+1}(X), \quad 0 \leq Ix < 1 \]  

(1)

where the weighting functions \( w(x) \) used to average (blend) the two adjacent preliminary local approximations \( \{F_I(X), F_{I+1}(X)\} \) are as yet un-specified. We prefer that the preliminary approximations \( \{F_1(X), F_2(X), \ldots, F_K(X), \ldots\} \) be left completely arbitrary, so long as they are smooth and represent the local behavior of \( F(X) \) well. As developed in Reference [8], the weight function can be selected to guarantee that the averaged approximation \( \bar{F}(X) \) osculates with \( F_I(X) \) in value and first derivative as \( X \to Ix \), and likewise \( F(X) \) osculates with \( F_{I+1}(X) \) in value and first derivative as \( X \to Ix+1 \). Notice that the shifted weight functions add to unity, as they must for an unbiased estimate, e.g., \( w(Ix) + w(Ix - 1) = 1 \), or \( w(Ix - 1) = 1 - w(Ix) \) Observe that \( Ix+1 = Ix - 1 \), so \( 0 \leq Ix \leq 1, -1 \leq Ix+1 = Ix - 1 \leq 0 \). Notice also the first derivative of the average of Eq. (1) at an arbitrary point is

\[
\frac{d\bar{F}_I(X)}{dx} = w(Ix) \frac{dF_I(X)}{dx} + w(Ix+1) \frac{dF_{I+1}(X)}{dx} + \frac{dw(Ix)}{dx} F_I(X) + \frac{dw(Ix+1)}{dx} F_{I+1}(X)
\]

(2)

Thus the required weight function satisfies the following boundary conditions:

\[
\begin{align*}
\text{at } x = 0 & : \quad \begin{cases} w(0) = 1 \\ \frac{dw}{dx} |_{x=0} = 0 \end{cases} \quad \text{at } x = 1 : \quad \begin{cases} w(1) = 0 \\ \frac{dw}{dx} |_{x=1} = 0 \end{cases}
\end{align*}
\]

(3)

With these boundary conditions, the first term of Eq. (1) reduces to \( F_I(X) \) as \( Ix \to 0 \) and likewise, only the first term of Eq. (2) contributes as \( Ix \to 0 \). Analogous arguments hold at the right end of the interval. If polynomials of lowest degree are used to satisfy the boundary conditions of Eq. (3), then the weight function can be shown (Ref. [8]) to be simply:

\[
w(x) = \begin{cases} 1 - x^2(3 + 2x), & -1 \leq x < 0 \\ 1 - x^2(3 - 2x), & 0 \leq x \leq 1 \end{cases} = 1 - x^2(3 - 2|Ix|)
\]

(4)

These are the functions plotted in Figure 3.

In the event that discrete measurements of \( F(X) \) are available, the preliminary approximations \( \{F_1(X), F_2(X), \ldots, F_K(X), \ldots\} \) are fit to data subsets in the \( \Delta X = \pm h \) regions centered on \( \{1X, 2X, \ldots, kX, \ldots\} \). It is evident that the final approximation on each interval is the average of overlapping weighted least square approximations, fit to shifted data lying within \( \pm h \) of the vertices. For equally precise measurements of \( F(X) \), the least square process should use the same weight functions of Eq. (3). If the measurements are made with unequal expected precision, then the statistically justified weights should be scaled using the weights of Eq. (4). Note the qualitative justification: “If one least square fit is good, the average of two should be better.” Observe that simply through choosing the judicious weight functions of Eq. (4) we are guaranteed global piecewise continuity for all possible continuous local approximations \( \{F_1(X), F_2(X), \ldots, F_K(X), \ldots\} \). One retains the freedom to vary the degree
of the local approximations, as needed, to fit the local behavior of $F(X)$, and rely upon the weight functions to enforce continuity.

The generalization to 2-Dimensions is amazingly straightforward. Note that the local approximations $\{F_{11}(X_1, X_2), F_{12}(X_1, X_2), \cdots, F_{h_1h_2}(X_1, X_2), \cdots\}$ are valid over $(2h) \times (2h)$ regions centered on the vertices $\{(1X_1, 1X_2), (2X_1, 2X_2), \cdots, (h_1X_1, h_2X_2), \cdots\}$.

Given four contiguous vertices:
\[
\begin{align*}
( & h_1X_1, h_2+1X_2 = h_1X_2 + h) \\
( & h_1+1X_1 = h_1X_1 + h, \ h_2+1X_2 = h_2X_2 + h) \\
( & h_1+1X_1 = h_1X_1 + h, \ h_2X_2) \\
( & h_1X_1, h_2X_2)
\end{align*}
\]  

(5)

The corresponding four preliminary approximations are valid in the $(2h) \times (2h)$ regions centered at the contiguous four nodes are denoted:
\[
\begin{align*}
F_{l_1l_2+1}(X_1, X_2) & \quad F_{l_1+1l_2+1}(X_1, X_2) \\
F_{l_1l_2}(X_1, X_2) & \quad F_{l_1+1l_2}(X_1, X_2)
\end{align*}
\]  

(6)
The final averaged approximation valid within the $h \times h$ region bounded by the four vertices of Eq. (5) is given by

$$
\bar{F}_{I_1I_2}(X_1, X_2) = \frac{1}{i_1} \frac{1}{i_2} \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} w_{I_1I_2}(I_1+i_1X_1, I_2+i_2X_2) F_{I_1+i_1I_2+i_2}(X_1, X_2)
$$

(7)
where, it can be verified that choosing the weight functions as

\[ w_{ij}(x_1, x_2) = \begin{cases} 1 - x_1^2(3 + 2x_1) & x_1 < 0, \\ 1 - x_1^2(3 + 2x_1) & x_1 > 1 \end{cases} \]

(8)

then these functions are a partition of unity so that they satisfy

\[ \sum_{i=0}^{N} \sum_{j=0}^{N} w_{ij}(x_1, x_2) = 1 \]

(9)

as necessary for Eq. (7) to give an un-biased average. Note that if we use a common origin (the lower left vertex) for all four weight functions, then the one centered on the origin is:

\[ w_{0,0}(x_1, x_2) = |1 - x_1^2(3 + 2x_1)||1 - x_2^2(3 + 2x_2)| \]

(10)

The remaining three weight functions are simply obtained by translating this function to the other three vertices as:

\[ w_{1,0}(x_1, x_2) = w_{0,0}(x_1 - 1, x_2) \]
\[ w_{0,1}(x_1, x_2) = w_{0,0}(x_1, x_2 - 1) \]
\[ w_{1,1}(x_1, x_2) = w_{0,0}(x_1 - 1, x_2 - 1) \]

(11)

These four overlapping weight functions are shown in Figure 5. The central unit square of Figure 5 is the focus of this figure, it is the region in which the final averaged approximation of Eq. (7) is valid. The process can be shifted by one unit cell in any direction and continuity arguments will lead to the conclusion that the adjacent final averaged approximations match in value and both partial derivatives along their common boundaries.

We see the weight function of Figure 2, from Ref. [8, 4] is obtained to within the obvious notation changes. The reason for changing notations in the present paper is that the generalization to N-dimensions follows easily from the above pattern.

The generalization of Eqs. (7) and (8) are:

\[ \hat{F}_{l_1, \ldots, l_N}(X_1, \ldots, X_N) = \sum_{l_1=0}^{N} \sum_{l_2=0}^{N} \ldots \sum_{l_N=0}^{N} \{ w_{l_1, \ldots, l_N}(l_1+i_1 X_1, \ldots, l_N+i_N X_N) \} \]

(12)

and

\[ \{ w_{l_1, \ldots, l_N}(l_1+i_1 X_1, \ldots, l_N+i_N X_N) \} = \prod_{i=1}^{N} w(l_i+i_X) \]

(13)

(14)
The partition of unity constraint of Eq. (14) is required for an unbiased average in Eq. (12). Further, it can be verified that the unbiased average requirement of Eq. (14) is satisfied everywhere in the hypercube where averaged final approximation \( \tilde{F}_{I_1 \cdots I_N} (X_1, \cdots, X_N) \) of Eq. (12) is valid.

The above approximation approach, and minor variations of it, has been used in a wide variety of modeling problems, including mathematical modeling of topography, the earth’s gravity field, the focal plane distortions of star cameras, mod-
boundaries with adjacent unit hypercubes with their corresponding weighted averaged final approximations. The generalized weight functions that guarantee arbitrary order continuity are given in Table 1. Only the weight function centered at the origin is tabulated, the other $2^N - 1$ weight functions are obtained by simply shifting the function using the origin translations to the other $2^N - 1$ vertices of the hypercube, analogous to Eqs. (11), e.g., using the $2^N - 1$ origin translations:

$$\{(0,0,\ldots,0,0,1),(0,0,\ldots,0,1,0),\ldots,(1,1,\ldots,1,1,1)\}$$ (15)

The weight functions for the first three orders of continuity, for one and two dimensional approximation, are shown in Figure 6 and 7, respectively.

Figure 6: 1-D Weighting Functions for Various Degrees of Piecewise Continuity
Consider the approximation of a one variable function $F(X)$. Suppose we are using the weighting function method as illustrated in Figure 3. The preliminary approximations, while arbitrary, in particular could be chosen to minimize the least square criterion

$$J = \frac{1}{2} \int_{-1}^{1} w(x)|F(X) - F_l(X)|^2 dx$$

(16)

Furthermore, we consider the case that $F_l(X)$ is a linear combination of a prescribed set of linearly independent basis functions $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ as

$$F_l(X) = \sum_{i=0}^{n} a_i \phi_i(x); \quad X = lX + hx$$

(17)
<table>
<thead>
<tr>
<th>order of piecewise continuity</th>
<th>Weight Function:</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$w(x) = 1 - y$</td>
</tr>
<tr>
<td>1</td>
<td>$w(x) = 1 - y^2(3 - 2y)$</td>
</tr>
<tr>
<td>2</td>
<td>$w(x) = 1 - y^3(10 - 15y + 6y^2)$</td>
</tr>
<tr>
<td>3</td>
<td>$w(x) = 1 - y^4(35 - 84y + 70y^2 - 20y^3)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$m$</td>
<td>$w(x) = 1 - y^{m+1} { \frac{(2m+1)!(-1)^m}{(m!)^2} \sum_{k=0}^{m} \frac{(-1)^k}{2m-k+1} \left( \frac{m}{k} \right) y^{m-r} }$</td>
</tr>
</tbody>
</table>

The least square criterion, making use of Eq. (16) can be written as

$$J = J_0 - c^T a + \frac{1}{2} a^T M a$$

where

$$J_0 = \frac{1}{2} \int_{-1}^{1} w(x) F^2(x) dx \equiv \frac{1}{2} < F(x), F(x) >$$

$$c^T = \left\{ \begin{array}{c} \int_{-1}^{1} w(x) F(x) \phi_0(x) dx \\ \int_{-1}^{1} w(x) F(x) \phi_1(x) dx \\ \vdots \\ \int_{-1}^{1} w(x) F(x) \phi_n(x) dx \end{array} \right\}$$

$$M = \begin{bmatrix} \mu_{00} & \mu_{01} & \cdots & \mu_{0n} \\ \mu_{10} & \mu_{11} & \cdots & \mu_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n0} & \mu_{n1} & \cdots & \mu_{nn} \end{bmatrix} = M^T; \mu_{ij} = < \phi_i, \phi_j > = \int_{-1}^{1} w(x) \phi_i(x) \phi_j(x) dx$$

$$a = \left\{ a_0, a_1, \cdots, a_n \right\}^T$$

Observe that minimization of Eq.(18) gives the optimum (minimum integral least square fit error) coefficients as

$$a = M^{-1} c$$

While Eq. (23) holds for an arbitrary set of linearly independent basis functions, for the special case that the basis functions satisfy the orthogonality condition

$$< \phi_i(x), \phi_j(x) > \equiv \int_{-1}^{1} w(x) \phi_i(x) \phi_j(x) dx = k_i \delta_{ij}, \ k_i \Delta \mu_{ii} = \int_{-1}^{1} w(x) \phi_i^2(x) dx$$

---

**Table 1: Weight Functions for Higher Order Continuity**
the least square solution of Eq. (23), as a consequence of the diagonal $M$ matrix, simplifies to the simple uncoupled result:

$$a_i = \frac{\langle F(x), \phi_i(x) \rangle}{k_i}, \ i = 1, 2, \cdots, n$$

(25)

Thus, if we can construct basis functions orthogonal with respect to the particular weight functions of Table 1, we enjoy the usual advantages of approximation of orthogonal functions in a global/local approximation setting. We consider the special case of $m = 1$; the construction of the corresponding orthogonal basis functions requires the Gram-Schmidt process. Using the methods of the Appendix, it can be verified that the basis functions given in Table 2 satisfy the orthogonality conditions of Eq. (24). Note that $c_n$ in Table 2 is determined so that $\phi_n(x)$ satisfies the normalization, $|\phi_n(\pm 1)| = 1$. These functions are plotted in Figure 8.

Table 2: One Dimensional Basis Functions Orthogonal with respect to the weight Function $(x) = 1 - x^2(3 - 2|x|)$

<table>
<thead>
<tr>
<th>degree</th>
<th>Basis Functions, $\phi_j(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$x$</td>
</tr>
<tr>
<td>2</td>
<td>$(-2 + 15x^2)/13$</td>
</tr>
<tr>
<td>3</td>
<td>$(-9x + 28x^3)/19$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$n$</td>
<td>$\phi_n(x) = \frac{1}{c_n} \left[ x^n - \sum_{j=0}^{n-1} \frac{\langle x^n \phi_j(x) \rangle}{\langle \phi_j(x), \phi_j(x) \rangle} \phi_j(x) \right]$</td>
</tr>
</tbody>
</table>

**Two Dimensional Case**

Consider the approximation of a two variable function $F(X_1, X_2)$. The preliminary approximations $F_{IJ}(X_1, X_2)$, while arbitrary, in particular could be chosen to minimize the least square criterion

$$J = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} w(x_1, x_2) [F(X_1, X_2) - F_{IJ}(X_1, X_2)]^2 dx_1 dx_2$$

(26)

Furthermore, we consider the case that $F_{IJ}(X_1, X_2)$ is chosen as a linear combination of a prescribed set of linearly independent basis functions $\{\phi_{ij}(x)\}; \ i = 1, 2, \cdots n; j = 1, 2, \cdots n$ as

$$F_{IJ}(X_1, X_2) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} \phi_{ij}(x_1, x_2); \ X_1 = I X_1 + h x_1, \ X_2 = J X_2 + h x_2$$

(27)
In particular, consider the multiplicative structure for the weight function

\[ w(x_1, x_2) = [1 - x_1^2(3 - 2|x_1|)][1 - x_2^2(3 - 2|x_2|)] \] (28)

and basis functions

\[ \phi_{ij}(x_1, x_2) = \phi_i(x_1)\phi_j(x_2) \] (29)

where we choose the one-dimensional basis functions \( \phi_i(x) \) from Table 2 that are orthogonal with respect to \( w(x) = 1 - x^2(3 - 2|x|) \). Introducing the inner product notation:

\[ <\alpha(x_1, x_2), \beta(x_1, x_2)> = \int_{-1}^{1} \int_{-1}^{1} w(\xi_1, \xi_2)\alpha(\xi_1, \xi_2)\beta(\xi_1, \xi_2)d\xi_1 d\xi_2 \] (30)

As a consequence of the orthogonality \( \phi_i(x) \) from Table 2, the choice of Eqs. (28), (29) and the definition, it is evident that the functions of Eqs. (28) are orthogonal,
and basis functions

In particular, consider the continued product structure for the weight function

Furthermore, we consider the case that

Consider the approximation of a function $F(x_1, x_2)$ of $N$ variables. The preliminary approximations $F_{i_1...i_N}(x_1, x_2, \ldots, x_N)$, while arbitrary, in particular could be chosen to minimize the least square criterion

As a consequence of orthogonality, it follows that the least square amplitudes are:

The first five sets (degrees zero through four) of the two dimensional orthogonal polynomials of Eq. (29) are shown in Figure 9.

**N-Dimensional Case**

Consider the approximation of a function $F(x_1, x_2, \ldots, x_N)$ of $N$ variables. The preliminary approximations $F_{i_1...i_N}(x_1, x_2, \ldots, x_N)$, while arbitrary, in particular could be chosen to minimize the least square criterion

Furthermore, we consider the case that $F_{i_1...i_N}(x_1, x_2, \ldots, x_N)$ is chosen as a linear combination of a prescribed set of linearly independent basis functions $\{\phi_{i_1...i_N}(x_1, \ldots, x_N)\}$ as

where,

In particular, consider the continued product structure for the weight function

and basis functions

because

$$
\langle \phi_{i_1}(x_1, x_2), \phi_{i_2}(x_1, x_2) \rangle = \int_{-1}^{1} \int_{-1}^{1} w(\xi_1, \xi_2) \phi_{i_1}(\xi_1, \xi_2) \phi_{i_2}(\xi_1, \xi_2) d\xi_1 d\xi_2
$$

$$
= \frac{1}{\sqrt{\pi}} \frac{1}{\kappa_{i_1} \kappa_{i_2}} \delta_{i_1 i_2}
$$

(31)

$$
a_{i_1 i_2} = \frac{\phi_{i_1}(x_1, x_2), F(x_1, x_2)}{\phi_{i_2}(x_1, x_2), \phi_{i_2}(x_1, x_2)} = \frac{\phi_{i_1}(x_1, x_2), F(x_1, x_2)}{k_i k_j}
$$

(32)

$$
J = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} w(x_1, \ldots, x_N) \left[ F(x_1, \ldots, x_N) - F_{i_1...i_N}(x_1, \ldots, x_N) \right]^2 dx_1 \cdots dx_N
$$

(33)

$$
F_{i_1...i_N}(x_1, \ldots, x_N) = \sum_{i_1=0}^{n} \cdots \sum_{i_N=0}^{n} a_{i_1...i_N} \phi_{i_1...i_N}(x_1, \ldots, x_N)
$$

(34)

$$
X_i = i_1 x_i + h x_i, \ldots, X_N = i_N x_N + h x_N
$$

(35)

$$
w(x_1, \ldots, x_N) = [1 - x_1^2(3 - 2|x_1|)] \cdots [1 - x_N^2(3 - 2|x_N|)] = \prod_{i=0}^{N} [1 - x_i^2(3 - 2|x_i|)]
$$

(36)

$$
\phi_{i_1...i_N}(x_1, \ldots, x_N) = \prod_{i=0}^{N} \phi_i(x_i)
$$

(37)
where we choose the one-dimensional basis functions $\phi_i(x)$ from Table 2 that are orthogonal with respect to $w(x) = 1 - x^2(3 - 2|x|)$.

Introducing the inner product notation:

$$<\alpha(x_1, \ldots, x_N), \beta(x_1, \ldots, x_N)> \Delta \int_{-1}^{1} \int_{-1}^{1} w(\xi_1, \ldots, \xi_N) \alpha(\xi_1, \ldots, \xi_N) \beta(\xi_1, \ldots, \xi_N) d\xi_1 \cdots d\xi_N$$

As a consequence of the orthogonality $\phi_i(x)$ from Table 2, the choice of Eqs. (33), (36), and (37) it is evident that the functions of Eqs. (34) are orthogonal, because

$$<\phi_{i_1 \cdots i_N}(x_1, \ldots, x_N), \phi_{j_1 \cdots j_N}(x_1, \ldots, x_N)>$$

$$= \int_{-1}^{1} \int_{-1}^{1} (w(\xi_1) \cdots w(\xi_N) \phi_{i_1 \cdots i_N}(\xi_1, \ldots, \xi_N) \phi_{j_1 \cdots j_N}(\xi_1, \ldots, \xi_N)) d\xi_1 \cdots d\xi_N$$

$$= [k_{i_1} k_{i_2} \cdots k_{i_N}] [\delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_N j_N}]$$

(39)
As a consequence of orthogonality, it follows that the least square amplitudes are:

\[ a_{i_1 \cdots i_N} = \frac{\langle \phi_{i_1 \cdots i_N}(x_1, \cdots, x_N), F(x_1, \cdots, x_N) \rangle}{\prod_{j=1}^{N} k_j} \]

Remarkably, the weight functions, basis functions, and orthogonality conditions are generated directly from the one dimensional results. Furthermore, we arrive with a piecewise continuous approximation in \( N \) dimensions with the added benefits that the local approximations are linear combinations of basis functions orthogonal to the weight function used in averaging the overlapping approximations.

**Illustrative Examples**

The approximation algorithm presented in this paper was tested on a variety of test functions and experimental data obtained by wind tunnel testing of synthetic jet actuation wing. In this section, we present some results from the studies, importantly a test case for function approximation and a dynamical System identification from wind tunnel testing of synthetic jet actuation wing. Also, a vibrating Space Based Radar (SBR) antenna surface approximation is considered.

**Function Approximation**

The test case for the function approximation is constructed using by the following analytic surface function [7].

\[
f(X_1, X_2) = \frac{10}{(X_2 - X_1^2)^2 + (1 - X_1)^2 + 1} + \frac{5}{(X_2 - 8)^2 + (5 - X_1)^2 + 1} + \frac{5}{(X_2 - 8)^2 + (8 - X_1)^2 + 1}
\]

A random sampling of the interval \([0 - 10, 0 - 10]\) for \( X_1 \) and \( X_2 \) is used to obtain 60 samples of each. Figures 10(b) shows the true surface plot of the training set data points. According to our experience this particular function has many important features such as sharp ridge line that is very difficult to learn with many existing function approximation algorithms with reasonable number of nodes.

To approximate the function given in equation (40), we divide the whole input region into a set of finite element cells, defined with cartesian coordinates, \( X_1 \) and \( X_2 \). Therefore, \( 10 \times 10 \) modeling region can be divided into different number of cells depending upon cell length. For example, the whole input region can be modeled by a total of 16 cells of dimension 1.2766 \times 1.2766 or by a total of 576 cells of dimension 0.4082 \times 0.4082. Further, a grid of all function observations is generated from measurement data and a continuous approximation for a particular cell is generated via a least-square procedure.
The local approximation of analytical function $\hat{f}(x_1, x_2)$, for a particular cell is modeled by orthogonal polynomials of the form:

$$\hat{f}(x_1, x_2) = \sum_i \sum_j a_{ij} \phi_i(x_1, x_2) \phi_j(x_1, x_2), \quad i + j \leq 2 \quad (41)$$

The orthogonal functions (listed in table 2), $\phi_i$ and $\phi_j$, are chosen in such a way that the degree of $\hat{f}(x_1, x_2)$ is always less than or equal to 2. Further, $x_1$ and $x_2$ denote the local cell coordinates defined as below:

$$x_1 = 2(X_1 - X_{1m})/X_{1cell} \quad x_2 = 2(X_2 - X_{2m})/X_{2cell} \quad (42)$$

where, $(X_{1m}, X_{2m})$ and $X_{1cell} \times X_{2cell}$ represent the centroid and dimension of a particular cell respectively.

As $\phi_i$ and $\phi_j$ are chosen to be orthogonal polynomial functions therefore, unknown coefficients $a_{ij}$ can be determined from equation (32). For a discrete observation grid the inner product in equation (32) is replaced by following equation:

$$<\phi_{ij}(x_1, x_2), \phi_{kl}(x_1, x_2)> = \sum_{m=1}^{N} w(x_{1m}, x_{2m}) \phi_{ij}(x_{1m}, x_{2m}) \phi_{kl}(x_{1m}, x_{2m}) \quad (43)$$

where, $N$ is the total number of observation points. Now, due to discrete observations the orthogonal condition of equation (31) can break down i.e. $<\phi_{ij}(x_1, x_2), \phi_{kl}(x_1, x_2)> \neq k_i \delta_{ik} k_j \delta_{jl}$. However, we should mention that as $N \to \infty$ the orthogonality error for discrete observation decreases to zero. Figure 10(a) shows the plot of discrete orthogonalization error, defined as $||<\phi_{ij}(x_1, x_2), \phi_{kl}(x_1, x_2)> - k_i \delta_{ik} k_j \delta_{jl}||$, versus number of observation points in a particular cell. From this plot, it is clear that orthogonalization error decreases as number of points inside a particular cell increases. In a future paper, we will present a procedure to generate discrete orthogonal polynomials of many variables.

As discussed earlier the approximation error depends upon the grid size, therefore, it was decided to study the root mean square approximation error as a function of cell size for a fixed order of polynomials. Figure 10(e) shows the plot of root mean square error versus cell size. From this figure, it is apparent that root mean square error shows a minimum value for a particular cell size. It is due to the fact that approximation error depends upon cell size and total number of observations in a particular cell. To achieve better approximation accuracy, we would like to have more observations in a particular cell and simultaneously would like to reduce the size of cell to capture the local behavior of the unknown function. However, for a fixed number of total observations, the number of observations in a particular cell decreases as cell size is reduced and increases as cell size is increased. It is due to this reason, approximation error is minimum for a particular size of finite element cell. Finally, figures 10(c) and 10(d) show surface plots of approximated surface and relative error surface respectively. These results corresponds to total 256 cells of length 0.6186. The relative percentage approximation error is computed at cell...
Figure 10: Local Polynomial Approximation Results for May-West Function
centroid using following formula:

\[ \text{Relative error} = \frac{f_{\text{approx}} - f_{\text{true}}}{f_{\text{true}}} \times 100 \]  

(44)

From these figures, it is clear that we are able to learn the analytical function given in equation (40) very well with relative approximation errors less than 1%.

**Synthetic Jet Actuator Modeling**

In this section, the local polynomial approximation results for the synthetic jet actuator are presented. These results show the effectiveness of the local polynomial approximation algorithm presented in this paper to learn the input-output mapping for the synthetic jet actuation wing.

**Experimental Set up**

A Hingeless-Control-Dedicated experimental setup has been developed, as part of the initial effort, the heart of which is a stand-alone control unit, that controls all of the wing’s and SJA’s parameters and variables. The setup is installed in the 3’ × 4’ wind tunnel of the Texas A&M Aerospace Engineering Department (Figure 11). The test wing profile for the dynamic pitch test of the synthetic jet actuator is a NACA 0015 airfoil. This shape was chosen due to the ease with which the wing could be manufactured and the available interior space for accommodating the synthetic jet actuator (SJA).

Experimental evidence suggests that a SJA, mounted such that its jet exit tangentially to the surface, has minimal effect on the global wing aerodynamics at low to moderate angles of attack. The primary effect of the jet is at high angles of attack when separation is present over the upper wing surface. In this case, the increased mixing associated with the action of a synthetic jet, delays or suppresses flow separation. As such, the effect of the actuator is in the non-linear post stall domain. To learn this nonlinear nature of SJA experiments were conducted with the control-dedicated setup shown in figure 11. The wing angle of attack (AOA) is controlled by the following reference signal.

1. Oscillation type: sinusoidal Oscillation magnitude: 12.5°.
2. Oscillation offset (mean AOA): 12.5°
3. Oscillation frequency: from 0.2Hz to 20Hz.

In other words, the AOA of airfoil is forced to oscillate from 0° to 25° at a given frequency (see figure 12). The experimental data collected were the time histories of the pressure distribution on the wing surface (at 32 locations). The data was also integrated to generate the time histories of the lift coefficient and the pitching moment.
Data was collected with the SJA on and with the SJA off (i.e. with and without active flow control). All the experimental data were taken for 5 sec at a 100 Hz sampling rate. The experiments described above were performed at a freestream velocity of \(25 \text{ m/sec}\). From the surface pressure measurements, the lift and pitching moment coefficients were calculated via integration. As the unknown SJA model is known to be dynamic in nature so SJA wing lift force and pitching moment coefficients are modelled by second order system i.e. they are assumed to be function of current and previous time states (angle of attack).

\[
C_L(t_k) = C_L(\alpha_k, C_L(t_{k-1})) \quad (45)
\]

\[
C_M(t_k) = C_M(\alpha_k, C_M(t_{k-1})) \quad (46)
\]

Like in the function approximation example, the moment and lift data is grided based upon the time interval. To approximate the dynamics in a particular time interval the
(a) Angle of Attack Variation without SJA.  
(b) Angle of Attack Variation with SJA actuation frequency of 60 Hz.

Figure 12: Angle of Attack Variation.

Figure 13(a) shows the plot of root mean square lift force approximation error versus the size of the local time interval. From this figure, it is clear that the approximation error increases as the size of the local time interval increases. We mention that if time interval size is less than 25 then we face the problem of ill-conditioned matrix as observation points are pretty close to each other.

Figures 13(b) and 13(c) show the measured and approximated lift coefficient for zero and 60 Hz jet actuation frequency respectively with time interval size of 25. Figures 13(d) and 13(e) show the corresponding approximation error plots. From these figures, it is clear that we are able to learn the nonlinear relationship between lift coefficient and angle of attack with and without SJA on.

Similarly, figures 14(a) and 14(b) show the measured and RBFN approximated pitching moment coefficient for zero and 60 Hz jet actuation frequency respectively. Figures 14(c) and 14(d) show the corresponding approximation error plots. From these figures, it is clear that we are able to learn the nonlinear relationship between moment coefficient and angle of attack (with and without SJA on) very well within experimental accuracy.
Figure 13: Lift Force Approximation Results
Space Based Radar (SBR) Antenna Simulation

Space Based Radar systems envisioned for the future may be a constellation of spacecraft that provide persistent real-time information of ground activities through the identification and tracking of moving targets, high-resolution synthetic aperture radar imaging, and collection of high-resolution terrain information. The accuracy of the information obtain from SBR systems depend upon many parameters like the geometric shape of the antenna, permittivities of the media through which radar wave is traveling, etc. Therefore the characteristics of the scattered wave received by the SBR antenna for a given frequency depend on the surface and geometric parameters of the radar. Therefore, to apply necessary corrections for scattering of radar waves the precise knowledge of SBR antenna becomes a necessity. However, excitation of flexible dynamics mode makes shape estimation problem a bit difficult. While a variety of surface models can be employed to model the instantaneous shape, we consider the case that the surface is measured at discrete points and a smooth least square model is
The objective of this section is to apply the local polynomial approximation methodology, developed in this paper, to estimate the real time SBR antenna shape. For simulation purposes, the SBR antenna (shown in Fig. 15) is assumed to be 20 m long in length and have a constant cross section along the length given by following undeformed surface shape:

\[ X = \frac{3 \cos \theta_0}{1 + 1.9 \sin \theta_0}, \quad Y = \frac{3 \sin \theta_0}{1 + 1.9 \sin \theta_0}, \quad 0 \leq \theta_0 \leq \frac{3\pi}{4} \] (48)

To construct the shape of antenna it is assumed that measurements of 25 points are available along a given cross section with the help of some vision sensor. Further, such 20 cross section measurements are assumed to be available along the length of the antenna at a particular time with a sampling frequency of 10 Hz. Further, true measurements are corrupted by Gaussian white noise of standard deviation of 1 mm. To make the shape estimation problem more interesting the shape of antenna is assumed to vary with respect to both in spatial coordinates and time according to following equations:

\[ X = r \cos \theta, \quad Y = r \sin \theta \] (49)

where, \( r = 3 + 0.4 \cos(5\pi Z) \sin(5t) \) and \( \theta = \theta_0 + \frac{\pi}{50} \cos(\pi Z) \sin(10t) \).

Further, the expression for \( r \) and \( \theta \) can be assumed to depict the bending and torsion mode for SBR antenna. However, this motion is just representative and may be a poor approximation of the actual flexible dynamics. We mention that in this paper, we do not worry about the actual flexible dynamics of the SBR antenna, as our purpose is just to demonstrate the local approximation methodology developed in this paper.

To approximate the SBR antenna shape at a particular time, the measurement data is gridded using a set of finite element cells of size \( 2 \times 2 \times 3 \) m, defined with cartesian coordinates, \( X, Y \) and \( Z \). Now, a continuous approximation of SBR antenna shape, for a particular cell is generated via a least-square procedure as listed in section.

The SBR antenna shape for a particular cell is modeled by orthogonal polynomials given in Table 2:

\[ \hat{x}(x,y,z) = \sum_{i} \sum_{j} \sum_{k} a_{ijk}\phi_i(x)\phi_j(y)\phi_k(z), \quad i + j + k \leq 2 \] (50)

\[ \hat{y}(x,y,z) = \sum_{l} \sum_{m} \sum_{n} a_{lmn}\phi_l(x)\phi_m(y)\phi_n(z), \quad l + m + n \leq 2 \] (51)

It should be noticed that, \( x, y, \) and \( z \) denote the local cell coordinates defined as below:

\[ x = \frac{2(X - X_m)}{X_{cell}} \quad y = \frac{2(Y - Y_m)}{Y_{cell}} \quad z = \frac{2(Z - Z_m)}{Z_{cell}} \] (53)

where, \( (X_m,Y_m,Z_m) \) and \( X_{cell} \times Y_{cell} \times Z_{cell} \) represent the centroid and dimension of a particular cell respectively.

Figure 16 shows the true and approximated shape of the SBR antenna at different time intervals. While, Figure 17 shows the contour plots for the difference between...
Figure 15: Nominal SBR Antenna Shape

(a) Time, t=2 seconds

(b) Time, t=8 seconds

(c) Time, t=16 seconds

(d) Time, t=20 seconds

Figure 16: Space Based Radar Antenna Simulation Results: True and Approximated Surface Plots.
nominal and instantaneous antenna shape. From these figures, it is clear that mean RMS approximation error for X and Y coordinate are even less than half a percent at all time intervals. Therefore, we can conclude that we are able to learn the SBR antenna shape precisely even in presence of measurement errors. However, we should mention that approximation accuracy will depend upon the order of polynomials used as well as the excited flexible dynamics mode.

Concluding Remarks

We have presented a general methodology for input/output mapping in N dimensions. The method averages overlapping local preliminary approximations whose centroids of validity lie at the vertices of a user specified N dimensional grid. For simplicity
in this paper, the grid was taken as uniform, however the grid is more generally non uniform. The averaging makes use of a special class of weight functions that guarantee a prescribed degree of piecewise continuity and osculation with the preliminary approximations at their centroids of validity. The preliminary approximations can be chosen arbitrarily to take advantage of prior knowledge of a particular problem. Alternatively, the preliminary approximations can be chosen as linear combinations of any complete set of linearly independent basis functions. A particularly attractive choice is shown to be polynomial basis functions that are orthogonal with respect to the weight functions of the averaging process. We constructed these new orthogonal polynomials using a Gramm Schmidt process. The result is an new method for orthogonal function local approximation with an associated averaging process giving a global piecewise continuous approximation. The broad generality of the method, together with a number of examples presented provides a strong basis for optimism for the importance and utility of these ideas.

Appendix

*Gram-Schmidt Procedure of Orthogonalization*

Let \( \mathcal{V} \) be a finite dimensional inner product space spanned by basis vector functions \( \{w_1, w_2, \cdots, w_n\} \). According to the *Gram-Schmidt Process* an orthogonal set of basis functions \( \{v_1, v_2, \cdots, v_n\} \) can be constructed from any basis functions \( \{w_1, w_2, \cdots, w_n\} \) by following three steps:

1. Initially there is no constraining condition on the first basis element \( v_1 \) therefore we can choose \( v_1 = w_1 \).

2. The second basis vector, orthogonal to the first one, can be constructed by satisfying the following condition:
   \[
   < v_2, v_1 > = 0
   \]  
   (54)

   Further, if we write:
   \[
   v_2 = w_2 - cv_1
   \]  
   (55)

   then we can determine the following value of unknown scalar constant \( c \) by substituting this expression for \( v_2 \) in orthogonality condition, given by equation (54):
   \[
   c = \frac{< w_2, v_1 >}{< v_1, v_1 >}
   \]  
   (56)

3. Continuing the procedure listed in step 2, we can write \( v_k \) as:
   \[
   v_k = w_k - c_1 v_1 - c_2 v_2 - \cdots - c_{k-1} v_{k-1}
   \]  
   (57)
where, the unknown constants $c_1, c_2, \cdots, c_{k-1}$ can be determined by satisfying following orthogonality conditions:

$$< v_k, v_j > = 0 \quad \text{For } j = 1, 2, \cdots, k - 1$$

(58)

Since, $v_1, v_2, \cdots, v_{k-1}$ are already orthogonal to each other therefore the scalar constant $c_j$ can be written as:

$$c_j = \frac{< w_k, v_j >}{< v_j, v_j >}$$

(59)

Therefore, finally we have following general *Gram-Schmidt formula* for constructing the orthogonal basis vectors $v_1, v_2, \cdots, v_n$:

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{< w_k, v_j >}{< v_j, v_j >} v_j, \quad \text{For } k = 1, 2, \cdots, n$$

(60)

To construct the orthogonal polynomials of degree $\leq n$ with respect to weight function, $1 - x^2 (3 - 2|x|)$ on closed interval $[-1, 1]$, we need to apply the Gram-Schmidt procedure to non-orthogonal monomial basis $1, x, x^2, \cdots, x^n$. First of all, we compute the general expression for $< x^k, x^l >$:

$$< x^k, x^l > = \int_{-1}^{1} x^{k+l} (1 - x^2 (3 - 2|x|)) dx = \begin{cases} \frac{2}{k+l+1} - \frac{6}{k+l+3} + \frac{4}{k+l+2} & k + l \text{ is even} \\ 0 & k + l \text{ is odd} \end{cases}$$

(61)

According to this formula, monomials of odd degree are orthogonal to monomials of even degree. Now, if $p_0(x), p_1(x), \cdots$ denote the resulting orthogonal polynomials then we can begin the process of Gram-Schmidt orthogonalization by letting:

$$p_0(x) = 1$$

(62)

According to equation (60), the next orthogonal polynomial is

$$p_1(x) = x - \frac{< x, p_0 >}{< p_0, p_0 >} p_0(x) = x$$

(63)

Further, recursively using the Gram-Schmidt formula given by equation (60), we can generate the orthogonal polynomials given in Table 2, including the recursive form given for $\phi_n(x)$.

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