Desired Order Continuous Polynomial Time Window Functions for Harmonic Analysis

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Abstract

An approach for the construction of a family of desired order continuous polynomial time window functions is presented without self convolution of the parent window. The higher order of continuity of time window functions at the boundary of observation window helps in suppressing the spectral leakage. Closed-form expression for window functions in time domain and their corresponding Fourier transform are derived. The efficacy of these new window functions in discerning weak signal is demonstrated by computer simulations.

I. INTRODUCTION

Fast Fourier Transform (FFT) is widely used to measure the frequency content of sampled measurement data. The FFT transform analysis is based upon a finite sampled data set and assumes circular topologies for both the time domain and frequency domain. In other words, two endpoints of the time waveform are interpreted as though they were connected together. As a consequence of this, the truncated waveform exhibit different spectral characteristics from the original continuous-time signal. Time window functions are used to obtain a continuous waveform without any discontinuities and hence minimize this effect known as spectral leakage. Time window functions shape the finite time sampled measurement data, to minimize edge effects that result in spectral leakage in the FFT spectrum. Time window functions play an important role in digital signal processing, system identification, digital filter design and in other applications.

A detailed analysis and comparison of different window functions is presented in the seminal work of Harris [1]. Harris introduced a number of figures of merit to evaluate the spectral leakage error, allowing objective comparison between different window functions. Harris and later Geçkinli [2] discussed that spectral leakage can be suppressed by reducing the order of discontinuity at the boundary of observation windows. The discontinuity of the windowed signal can be avoided by matching as many orders of

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derivatives (of the weighted data) as possible at the boundary which is equivalent to setting the values of derivatives of the window functions to zero. This is generally achieved by self-convolving a parent window function multiple times in the time-domain [1]. For instance, a $C^m$-continuous time window function can be obtained by applying $m$ time convolution of the $C^0$-continuous rectangular window. Although the window function generated by $m$-fold self convolution of a parent window of length $N$ may exhibit better performance in suppressing spectral leakage than the parent window function, it will also increase the length of the resulting window to approximately $mN$. Furthermore, the Fourier transform of the resulting window function will have $m$ repeated zeros at each location where the parent window transform had a zero. Ideally, these additional zeros can be better placed between existing zeros to further reduce the sidelobes and better detect weak signals [1].

Furthermore, in Refs. [3]–[8], the effect of windowing on signal-to-noise ratio, harmonic distortions, multifrequency parameter estimation and quasi-synchronous sampling are studied in detail. It has been concluded that no window function is the best in all aspect and one should select one according to the requirement of a particular application. For example, Hanning window function is considered useful for noisy measurements while Kaiser-Bessel is recommended to detect two tones with frequencies very close to each other but with very different amplitudes. Over time, many different window functions [9]–[14] have been derived to optimize some features of a window function and easy implementation. In Ref. [9], new window functions have been derived with very good sidelobe behavior whereas in Ref. [10] time window functions are derived from B-splines that have fast sidelobe decay and maximum variance in the time domain. In Ref. [11], extremely flat top window functions are derived by multiple time-convolution of weighted-cosine windows with parameters optimized for flatness of the main lobe. Ref. [12]–[14] present window functions derived from hyperbolic functions and amplitude shaping pulses.

In this paper, we present an approach for the construction of a family of polynomial time window functions which allow desired order of continuity at the boundary of observation windows. It is well-known that if the $m^{th}$ derivative of a window function is discontinuous, then the sidelobes of window functions decays asymptotically as $6m$ dB/oct [1], [2]. The freedom to choose window continuity at the boundary of observation window allows us to tradeoff between different merits of the window functions. Another advantage of these window functions is that all their coefficients are integers and thus, they are easy to evaluate. The structure of paper is as follows: first, a closed-form expression for a desired order
continuous polynomial window functions in time domain is derived followed by the Fourier transform of these functions. Further, these window functions are compared with conventional window functions on the basis of figures of merit (FOM) like Equivalent Noise Bandwidth (ENBW), Sidelobe Fall Off, Coherent Gain and Scallop Loss.

II. POLYNOMIAL TIME DOMAIN WINDOW FUNCTIONS

In this section, we will present an approach for the construction of polynomial window functions in time domain which allow desired order of continuity at the boundary of the observation window. Without any loss of generality, we assume the window interval to be $[-1, 1]$. Let us assume that $f(t)$ represents the signal of interest and $\bar{f}(t)$ is the windowed approximation of $f(t)$ over the windowed interval $[-1, 1]$.

$$\bar{f}(t) = w(t)f(t) \tag{1}$$

where $w(t)$ is the window function with a compact support over $[-1, 1]$. The requirement that the windowed approximation $\bar{f}(t)$ in Eq. (1) form an $m^{th}$-order continuous signal leads to the following requirements for the necessary window functions:

1) The first derivative of the window function must have $m$ repeated zeros at the centroid of the windowed interval, i.e., at $t = 0$.

$$w(0) = 1, \quad \frac{d^k w}{dt^k} \bigg|_{t=0} = 0 \quad k = 1, \cdots, m \tag{2}$$

2) The window function must have an $(m+1)^{th}$-order zero at the end points of the windowed interval.

$$w(1) = 0, \quad \frac{d^k w}{dt^k} \bigg|_{t=1} = 0 \quad k = 1, \cdots, m \tag{3}$$

These aforementioned conditions are sufficient to ensure that the value and first $m$ time derivatives of the windowed signal $\bar{f}(t)$ reduces exactly to the first $m$ time derivatives of the original signal $f(t)$ at the centroid and end points of the windowed interval. In our prior work [15], a similar boundary value problem is solved to generate special weight function to blend two completely independent adjacent local approximations to a globally valid function. We use the same analysis here to derive a generic expression for an $m^{th}$-order continuous window function.
Like in Ref. [15], we assume that the window function is symmetric around the centroid of the windowed interval \([-1, 1]\), i.e., origin and develop the expression for the window function valid over the interval \([0, 1]\). In this respect, we assume the following particular form for the window function:

\[
    w(t) = 1 - u(t)
\]  

(4)

where \(u(t)\) is a polynomial function selected in such a way that the first \(m\) time derivatives of the window function vanish at centroid and end points.

\[
    \frac{d u(t)}{dt} = Ct^m (1 - t)^m
\]  

(5)

where \(C\) is a yet to be defined constant. Now the remaining boundary conditions, i.e., \(w(1) = 0\) can be used to find the constant \(C\).

\[
    w(1) = 1 - C \int_{0}^{1} \tau^m (1 - \tau)^m d\tau = 0
\]  

(6)

Now, making use of the fact that the integral expression, \(\int_{0}^{1} \tau^m (1 - \tau)^m d\tau = \frac{(m!)^2}{(2m+1)!}\), is a Eulerian integral of the first kind leads to the following value for the constant \(C\):

\[
    C = \frac{(2m + 1)!}{(m!)^2}
\]  

(7)

Substituting for the value of \(C\) in Eq. (5) and substituting the resultant expression in Eq. (4) leads to the following expression for the window function:

\[
    w(t) = w_m(t) = 1 - \frac{(2m + 1)!}{(m!)^2} \int_{0}^{1} \tau^m (1 - \tau)^m d\tau
\]  

(8)

Notice that we use \(w_m(t)\) to explicitly indicate the dependence of the window function on the smoothness order \(m\). Furthermore, making use of the binomial theorem to expand the integrand, \(\int_{0}^{1} \tau^m (1 - \tau)^m d\tau\) leads to

\[
    w_m(t) = 1 - \frac{(2m + 1)!}{(m!)^2} \int_{0}^{1} \sum_{n=0}^{m} \binom{m}{n} t^m \tau^{m-n} (-1)^n d\tau = 1 - K_m \sum_{n=0}^{m} A_{m,n} t^{2m-n+1}, \quad m C_n = \frac{m!}{n!(m-n)!}
\]  

(9)
where \( K_m \) and \( A_{m,n} \) are given by the following expressions:

\[
K_m = \frac{(2m + 1)!(-1)^m}{(m!)^2}, \quad A_{m,n} = \frac{(-1)^n m C_n}{2m - n + 1}
\]

Finally, to obtain the expression for the window function in the interval \([-1, 1]\) instead of \([0, 1]\), the absolute value of \( t \) can be used as the independent variable rather than \( t \).

\[
w_m(t) = 1 - K_m \sum_{n=0}^{m} A_{m,n} |t|^{2m-n+1}, \quad -1 \leq t \leq 1
\]

It should be noted that the exponent of \( t \), i.e., \( 2m - n + 1 \) is always greater than zero. These special functions of Eq. (11) are an integral part of recently developed multi-resolution function approximation algorithm known as Global-Local Orthogonal Mapping (GLO-MAP) [15] which permit inclusion of independent local functions over a compact support.

**II.1. Fourier Transformation of Window Functions**

In this section, we compute the Fourier transformation of the window functions derived in the previous section.

Let us consider the Fourier transformation for the window function \( w_m(t) \):

\[
W_m(\omega) = \int_{-\infty}^{\infty} w_m(t)e^{-i\omega t} dt
\]

Substituting for the expression for \( w_m(t) \) leads to

\[
W_m(\omega) = \int_{-1}^{0} e^{-i\omega t} \left[ 1 - K_m \sum_{n=0}^{m} A_{m,n} (-t)^{2m-n+1} \right] dt + \int_{0}^{1} e^{-i\omega t} \left[ 1 - K_m \sum_{n=0}^{m} A_{m,n} t^{2m-n+1} \right] dt
\]

Notice that the first integral expression in Eq. (13) is equivalent to

\[
\int_{-1}^{0} e^{-i\omega t} \left[ 1 - K_m \sum_{n=0}^{m} A_{m,n} (-t)^{2m-n+1} \right] dt = -\int_{1}^{0} e^{i\omega t} \left[ 1 - K_m \sum_{n=0}^{m} A_{m,n} t^{2m-n+1} \right] dt
\]
This leads to

\[ W_m(\omega) = \int_{0}^{1} \left( e^{i\omega t} + e^{-i\omega t} \right) \left[ 1 - K_m \sum_{n=0}^{m} A_{m,n} t^{2m-n+1} \right] dt = 2 \int_{0}^{1} \cos \omega t \left[ 1 - K_m \sum_{n=0}^{m} A_{m,n} t^{2m-n+1} \right] dt \]

\[ = 2 \frac{\sin \omega}{\omega} - 2K_m \sum_{n=0}^{m} A_{m,n} \int_{0}^{1} t^{2m-n+1} \cos \omega t dt \]  

(15)

Now let us consider the integral expression in Eq. (15):

\[ I_{2m-n+1} = \int_{0}^{1} t^{2m-n+1} \cos \omega t dt = \frac{t^{2m-n+1} \sin \omega t}{\omega} \bigg|_{0}^{1} - \frac{2m-n+1}{\omega} \int_{0}^{1} t^{2m-n} \sin \omega t dt \]  

(16)

\[ = \frac{\sin \omega}{\omega} - \frac{2m-n+1}{\omega} \left[ -\frac{t^{2m-n} \cos \omega t}{\omega} \bigg|_{0}^{1} + \frac{2m-n}{\omega} \int_{0}^{1} t^{2m-n-1} \cos \omega t dt \right], \quad 2m-n > 0 \]

\[ = \frac{\sin \omega}{\omega} + \frac{(2m-n+1) \cos \omega}{\omega^2} - \frac{(2m-n)(2m-n+1)}{\omega^2} I_{2m-n-1}, \quad 2m-n > 0 \]  

(17)

Notice that the \(2m-n > 0\) constraint corresponds to exponent of \(t\) in \(I_{2m-n-1}\) being greater than or equal to zero. Hence, the recursion formula given by Eq. (17) is only valid for \(2m-n > 0\). Consequently, we require \(I_0\) and \(I_1\) to permit us to use the recursion formula given by Eq. (17). It is easy to see that \(I_0\) and \(I_1\) are given as:

\[ I_0 = \frac{\sin \omega}{\omega}, \quad I_1 = \frac{\cos \omega + \omega \sin \omega - 1}{\omega^2} \]  

(18)

Now substitution of Eq. (17) in Eq. (15) leads to

\[ W_m(\omega) = \left( 2 - 2K_m \sum_{n=0}^{m} A_{m,n} \right) \frac{\sin \omega}{\omega} - 2K_m \sum_{n=0}^{m} A_{m,n} (2m-n+1) \left[ \frac{\cos \omega}{\omega^2} - \frac{(2m-n)}{\omega^2} I_{2m-n-1} \right], \quad 2m-n > 0 \]  

(19)

Making use of the fact that \(w(1) = 0\) and \(\frac{dw}{dt} \bigg|_{t=1} = 0\) for \(m > 0\), we get:

\[ K_m \sum_{n=0}^{m} A_{m,n} = 1, \quad K_m \sum_{n=0}^{m} A_{m,n} (2m-n+1) = 0 \]  

(20)

Finally, substitution of identities of Eq. (20) in Eq. (19) leads to the following generic expression for the Fourier transform of the window function, \(w_m(t)\):

\[ W_m(\omega) = 2K_m \sum_{n=0}^{m} I_{2m-n-1} A_{m,n} \frac{(2m-n+1)(2m-n)}{\omega^2}, \quad m > 0 \]  

(21)
Table I lists the closed-form expressions for these window functions in time-domain and their corresponding Fourier transforms for different order of continuity $m$. Furthermore, window functions for first 4 order of continuity and their corresponding Fourier transforms are shown in Figs. 1(a) and 1(b), respectively. From Fig. 1(b), it is clear that sidelobe far from mainlobe rolloff rapidly with an increase in $m$. This is consistent with the expected $6(m + 1)$ dB/oct rolloff for window function with $(m + 1)^{th}$ order discontinuous derivative [1], [2]. This attribute is reflected in reduced spectral leakage. Fig. 1(c) and 1(d) illustrate the mainlobe and a few sidelobes of $W_m(\omega)$, respectively. It is apparent that the width of mainlobe does not change significantly with order of continuity $m$. It should be noted that only the curve corresponding to $m = 0$ exhibit repeated roots at all locations. This is evidenced by the fact that the slope of $W_m(\omega)$ is zero whenever $W_m(\omega) = 0$. This is due to the fact that the window function corresponding to $m = 0$ is identical to the Bartlett window which can be generated by self-convolution of rectangular window [1]. It is apparent from Fig. 1(d) that increasing the order of smoothness of the window function does not result in repeated roots. This is evidenced by the fact that the slope of $W_m(\omega)$ is discontinuous whenever $W_m(\omega) = 0$ for $m > 0$. 
\[
\begin{array}{|c|c|c|}
\hline
m & w_m(t) & W_m(\omega) \\
\hline
0 & 1 - |t| & \frac{2(1 - \cos \omega)}{\omega^2} \\
1 & 1 - t^2(3 - 2|t|) & \frac{12(2 - 2 \cos \omega - \omega \sin \omega)}{\omega^4} \\
2 & 1 - |t|^3(10 - 15|t| + 6t^2) & \frac{120(12 - \omega^2 + \omega^2 \cos \omega - 6\omega \sin \omega - 12 \cos \omega)}{\omega^6} \\
\vdots & \vdots & \vdots \\
m & w_m(t) = 1 - K_m \sum_{n=0}^{m} A_{m,n} |t|^{2m-n+1} & 2K_m \sum_{n=0}^{m} \mathcal{I}_{2m-n-1} A_{m,n} \frac{(2m-n+1)(2m-n)}{\omega^2} \\
\hline
\end{array}
\]

TABLE I
Window functions \(w_m(t)\) and their Fourier Transform \(W_m(\omega)\) for different order of continuity \(m\).

II.2. Performance Analysis

To further gain insight into the performance of these window functions, we compute the following figures of merit: incoherent power gain, coherent gain, scalloping loss and equivalent noise bandwidth (ENBW). ENBW determines the capability of a window function to extract signal amplitude from background noise and is defined as the width of DFT of a rectangular window with the same peak power gain that would accumulate the same noise power [1]. This can be easily calculated from time samples of window function \(w(nT)\):

\[
ENBW = N \frac{\sum_{n} w^2(nT)}{\left[\sum_{n} w(nT)\right]^2}
\]  

(22)

Notice that lower value for ENBW implies better signal extraction from background noise. Rectangular window has the best possible value for the ENBW equal to 1. All other window functions will have the ENBW greater than one. Since the window function attenuates the signal at interval ends, it reduces the overall signal power. As a consequence of this, the amplitude measured at the DFT bin is not the same as
the real amplitude of the signal’s frequency component at that frequency. This reduction in signal power is called the *coherent gain* (CG) and is defined as [1]:

\[
CG = \frac{1}{N} \sum_n w(nT)
\]  

(23)

Notice that for rectangular window the coherent gain is one while other for any other window function the coherent gain is reduced due to the window smoothly going to zero at the boundaries. The incoherent power gain (ICG) represents the accumulated noise power of the window and is computed by making use of the following relationship:

\[
ICG = \frac{1}{N} \sum_n w^2(nT)
\]  

(24)

Notice that ENBW equal to the ratio of the incoherent power gain to the square of the coherent power gain. Another significant figure of merit is the scalloping loss (SL) related to the minimum detectable signal in the worst case of noncoherent sampling. Scalloping Loss is the apparent attenuation of the measured value for a frequency component that falls exactly half way between DFT bins. It is defined as the ratio of the power gain for a signal frequency component located half way between DFT bins, to the Coherent Power Gain for a signal frequency component located exactly on the DFT bin [1]:

\[
SL = \frac{\left| \sum_n w(nT)e^{-j\pi n/N} \right|}{\sum_n w(nT)} = \frac{|W(0.5\omega_s/n)|}{W(0)}
\]  

(25)

Figs. 2(a), 2(b) and 2(c) show the plots of Coherent Gain (CG) vs. Incoherent Gain Power (ICG), CG vs. Scalloping Loss (SL) and SL vs. ENBW, respectively. As expected, scalloping loss decreases while incoherent gain and ENBW increases with an increase in the order of continuity $m$ of the window function. It is interesting to notice that coherent gain remains constant for various order of continuity. It is also apparent from these plots that performance of these window function is comparable to many conventional window functions and one can actually tradeoff between different performance criterion by simply changing the order of continuity $m$.

**III. Numerical Results**

In this section, two examples are considered to evaluate the performance of the proposed window functions. First, we consider the following benchmark signal $f(t)$ introduced in Ref. [12], [13] to test
Fig. 2. Figures of Merits for Window Functions

(a) Coherent Gain (CG) vs. Incoherent Gain Power (ICG)

(b) Coherent Gain (CG) vs. Scalloping Loss (SL)

(c) Scalloping Loss (SL) vs. ENBW

the efficacy of these new window functions in harmonic signal analysis.

\[
f(t) = \sin(20\pi t) + 0.01\sin(30\pi t) + 0.001\sin(42\pi t) + 0.00001\sin(60\pi t)
\]  

(26)

The signal is sampled with \( N = 1024 \) samples with a sampling frequency of 111Hz which is not an integer multiple of frequencies to be identified. Fig. 3(a) shows the plot of a part of the sampled signal. Fig. 3(b) shows the plot of the DFT of the windowed signal by applying the polynomial window functions
corresponding to first six orders of continuity while Figs. 3(c) and 3(d) show the plot of the DFT of
the windowed signal by applying conventional window functions available in Matlab signal processing
toolbox. They have been plotted in separate figures to avoid clutter. From these plots, it is apparent that
while many conventional window functions (such as rectangular, Kaiser, Bartlett, Hamming etc.) struggle
in identifying the presence of weak signal corresponding to frequency $30Hz$, the newly developed window
functions (except for $m = 0$) correctly identify the relative strength of all frequency tones. This clearly
indicate the ability of the new window functions to discern the presence of the weak signals. This is
anticipated since the scalloping loss decreases with the increase in the smoothness order $m$.

The second example corresponds to a chirp signal which generate broadband excitation. The chirp signal
is simulated in Matlab with frequency swept between $10Hz$ and $30Hz$ in a quadratic manner. The
chirp signal is sampled with $N = 1024$ samples at a sampling frequency of $101Hz$. Fig. 4 illustrates
spectrogram plots for new window functions corresponding to $m = 0, \cdots, 5$ and Hanning window which
is the default window used in Matlab spectrogram command. As expected, all the plots clearly show
the quadratic time variation of frequency of the signal. It is also clear that as the smoothness index $m$
increases, the spectral leakage rapidly decreases as opposed to the Hanning window. This is a reflection
of the rapid sidelobe rolloff of $6(m + 1)dB/oct$.

IV. CONCLUSIONS

Recently developed polynomial blending functions of any desired continuity and with compact support are
exploited as window functions for harmonic analysis. A closed form expression for the Fourier transform
of proposed polynomial window function is derived. An advantage of the proposed approach is that it
results in desired order continuous window functions without the self convolution of the parent window
function and thus, the Fourier transform of the resulting window functions do not exhibit repeated zeros
at same location of the $m = 0$ window. The proposed polynomial time window functions are compared
to traditional window functions using figures of merit such as coherent gain, incoherent gain power,
scalloping loss and ENBW to illustrate their benefits and their efficacy in discerning the weak signal is
illustrated through numerical example.
Fig. 3. Example: $f(t) = \sin(20\pi t) + 0.01 \sin(30\pi t) + 0.001 \sin(42\pi t) + 0.00001 \sin(60\pi t)$
Fig. 4. Example2: Spectrogram of Chirp Signal
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REFERENCES


